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Transport Equations in Incompressible URANS and LES

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1 The Transport Equation for the Reynolds Stresses

The filtered Navier-Stokes equation for $\bar{u}_i$ reads

$$
\begin{aligned}
\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_k}(\bar{u}_i \bar{u}_k) &= -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \nu \frac{\partial^2 \bar{u}_i}{\partial x_k \partial x_k} - \frac{\partial \tau^a_{ik}}{\partial x_k} - g_i \beta \bar{t} \\
\tau_{ik} = \overline{u_i u_k} - \bar{u}_i \bar{u}_k,\ \tau^a_{ik} &= \tau_{ik} - \frac{1}{3}\delta_{ik}\tau_{jj}
\end{aligned}
\tag{1}
$$

where τ_{ik} denotes modelled SGS stress or URANS stress. The SGS/URANS turbulent kinetic energy is defined as $k_T = 0.5\tau_{ii}$. Decompose $\bar{u}_i$ and $\bar{p}$ into a time-averaged (or ensemble-averaged) value and a resolved fluctuation as

$$
\begin{aligned}
&\bar{u}_i = U_i + \bar{u}'_i,\ \bar{p} = P + \bar{p}',\ \bar{t} = T + \bar{t}' \\
&U_i = \langle \bar{u}_i \rangle,\ P = \langle \bar{p} \rangle,\ T = \langle \bar{t} \rangle \\
&u_i = \bar{u}_i + u''_i = U_i + \bar{u}'_i + u''_i
\end{aligned}
\tag{2}
$$

where u''_i is the SGS fluctuation. Insert this in Eq. 1 so that

$$
\begin{aligned}
\frac{\partial \bar{u}'_i}{\partial t} + \frac{\partial}{\partial x_k}((U_i + \bar{u}'_i)(U_k + \bar{u}'_k)) = -\frac{1}{\rho}\frac{\partial (P + \bar{p}')}{\partial x_i} &+ \nu \frac{\partial^2 (U_i + \bar{u}'_i)}{\partial x_k \partial x_k} \\
&- \frac{\partial \tau^a_{ik}}{\partial x_k} - g_i \beta (T + \bar{t}')
\end{aligned}
\tag{3}
$$

Time (ensemble) averaging of Eq. 3 yields

$$
\frac{\partial}{\partial x_k}(U_i U_k) = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_k \partial x_k} - \frac{\partial}{\partial x_k}\left(\langle \bar{u}'_i \bar{u}'_k \rangle + \langle \tau^a_{ik} \rangle\right) - g_i \beta T
\tag{4}
$$

Now subtract Eq. 4 from Eq. 3

$$
\begin{aligned}
&\frac{\partial \bar{u}'_i}{\partial t} + \frac{\partial}{\partial x_k}(U_i \bar{u}'_k + U_k \bar{u}'_i + \bar{u}'_i \bar{u}'_k) = \\
-\frac{1}{\rho}\frac{\partial \bar{p}'}{\partial x_i} + \nu \frac{\partial^2 \bar{u}'_i}{\partial x_k \partial x_k} + \frac{\partial}{\partial x_k}&\left(\langle \bar{u}'_i \bar{u}'_k \rangle + \underbrace{\langle \tau^a \rangle_{ik} - \tau^a_{ik}}_{-\tau'^a_{ik}}\right) - g_i \beta \bar{t}'
\end{aligned}
\tag{5}
$$

Multiply Eq. 5 with $\bar{u}'_j$ and a corresponding equation for $\bar{u}'_j$ by $\bar{u}'_i$, add them together, and time (ensemble) average

$$
\begin{aligned}
\left\langle \bar{u}'_j \frac{\partial}{\partial x_k}(U_i \bar{u}'_k + U_k \bar{u}'_i + \bar{u}'_i \bar{u}'_k) \right\rangle + \left\langle \bar{u}'_i \frac{\partial}{\partial x_k}(U_j \bar{u}'_k + U_k \bar{u}'_j + \bar{u}'_k \bar{u}'_j) \right\rangle = \\
-\left\langle \frac{\bar{u}'_j}{\rho}\frac{\partial \bar{p}'}{\partial x_i} \right\rangle - \left\langle \frac{\bar{u}'_i}{\rho}\frac{\partial \bar{p}'}{\partial x_j} \right\rangle + \nu \left\langle \bar{u}'_j \frac{\partial^2 \bar{u}'_i}{\partial x_k \partial x_k} \right\rangle + \nu \left\langle \bar{u}'_i \frac{\partial^2 \bar{u}'_j}{\partial x_k \partial x_k} \right\rangle \\
-\left\langle \bar{u}'_j \frac{\partial \tau'^a_{ik}}{\partial x_k} \right\rangle - \left\langle \bar{u}'_i \frac{\partial \tau'^a_{jk}}{\partial x_k} \right\rangle - g_i \beta \left\langle \bar{u}'_j \bar{t}' \right\rangle - g_j \beta \left\langle \bar{u}'_i \bar{t}' \right\rangle
\end{aligned}
\tag{6}
$$

The two first lines correspond to the usual $\overline{u_i u_j}$ equation in conventional Reynolds decomposition. The two last terms on line 2 can be re-written as

$$\begin{aligned}\nu\frac{\partial}{\partial x_k}\left\langle \bar{u}'_j\frac{\partial \bar{u}'_i}{\partial x_k}\right\rangle + \nu\frac{\partial}{\partial x_k}\left\langle \bar{u}'_i\frac{\partial \bar{u}'_j}{\partial x_k}\right\rangle - 2\nu\left\langle \frac{\partial \bar{u}'_j}{\partial x_k}\frac{\partial \bar{u}'_i}{\partial x_k}\right\rangle \\ = \nu\frac{\partial^2}{\partial x_k \partial x_k}\left\langle \bar{u}'_i\bar{u}'_j\right\rangle - 2\nu\left\langle \frac{\partial \bar{u}'_i}{\partial x_k}\frac{\partial \bar{u}'_j}{\partial x_k}\right\rangle\end{aligned} \tag{7}$$

The two first terms on the last line in Eq. 6 can be rewritten as

$$-\left\langle \frac{\partial}{\partial x_k}(\bar{u}'_j\tau'^a_{ik})\right\rangle + \left\langle \tau'^a_{ik}\frac{\partial \bar{u}'_j}{\partial x_k}\right\rangle - \left\langle \frac{\partial}{\partial x_k}(\bar{u}'_i\tau'^a_{jk})\right\rangle + \left\langle \tau'^a_{jk}\frac{\partial \bar{u}'_i}{\partial x_k}\right\rangle \tag{8}$$

Finally, we can now write the transport equation for $\langle \bar{u}'_i\bar{u}'_j\rangle$ as

$$\begin{aligned}\frac{\partial}{\partial x_k}(U_k\langle \bar{u}'_i\bar{u}'_j\rangle) = -\langle \bar{u}'_i\bar{u}'_k\rangle\frac{\partial U_j}{\partial x_k} - \langle \bar{u}'_j\bar{u}'_k\rangle\frac{\partial U_i}{\partial x_k} - \frac{1}{\rho}\left\langle \bar{u}'_i\frac{\partial \bar{p}'}{\partial x_j}\right\rangle - \frac{1}{\rho}\left\langle \bar{u}'_j\frac{\partial \bar{p}'}{\partial x_i}\right\rangle + \\ -\frac{\partial}{\partial x_k}\langle \bar{u}'_i\bar{u}'_j\bar{u}'_k\rangle + \nu\frac{\partial^2}{\partial x_k\partial x_k}\langle \bar{u}'_i\bar{u}'_j\rangle - 2\nu\left\langle \frac{\partial \bar{u}'_i}{\partial x_k}\frac{\partial \bar{u}'_j}{\partial x_k}\right\rangle - g_i\beta\left\langle \bar{u}'_j\bar{t}'\right\rangle - g_j\beta\left\langle \bar{u}'_i\bar{t}'\right\rangle \\ -\left\langle \frac{\partial}{\partial x_k}(\bar{u}'_j\tau'^a_{ik})\right\rangle - \left\langle \frac{\partial}{\partial x_k}(\bar{u}'_i\tau'^a_{jk})\right\rangle \\ + \left\langle \tau'^a_{ik}\frac{\partial \bar{u}'_j}{\partial x_k}\right\rangle + \left\langle \tau'^a_{jk}\frac{\partial \bar{u}'_i}{\partial x_k}\right\rangle\end{aligned} \tag{9}$$

where the two last lines include all terms related to the SGS/URANS stresses. The third line represents diffusion transport by SGS/URANS stresses and the fourth line represents dissipation by SGS/URANS stresses. For an eddy-viscosity SGS/URANS model

$$\tau^a_{ij} = -2\nu_T\bar{s}_{ij},\ \ \bar{s}_{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) \tag{10}$$

1.1 Resolved turbulent kinetic energy $\langle \bar{k}\rangle$

Now we will derive the transport equation for the resolved turbulent kinetic energy $\langle k\rangle = \langle \bar{u}'_i\bar{u}'_i\rangle/2$. Take the trace of Eq. 9 and divide by two

$$\begin{aligned}\frac{\partial}{\partial x_j}(U_j\langle \bar{k}\rangle) = -\langle \bar{u}'_i\bar{u}'_j\rangle\frac{\partial U_i}{\partial x_j} - \frac{\partial}{\partial x_j}\left(\frac{1}{\rho}\left\langle \bar{u}'_j\bar{p}'\right\rangle + \frac{1}{2}\langle \bar{u}'_i\bar{u}'_i\bar{u}'_j\rangle\right) + \nu\frac{\partial^2\langle \bar{k}\rangle}{\partial x_j\partial x_j} \\ -\nu\left\langle \frac{\partial \bar{u}'_i}{\partial x_j}\frac{\partial \bar{u}'_i}{\partial x_j}\right\rangle - g_j\beta\left\langle \bar{u}'_j\bar{t}'\right\rangle - \left\langle \frac{\partial}{\partial x_j}(\bar{u}'_i\tau'^a_{ij})\right\rangle + \left\langle \tau'^a_{ij}\frac{\partial \bar{u}'_i}{\partial x_j}\right\rangle\end{aligned} \tag{11}$$

The pressure-velocity term was re-written as

$$\left\langle \bar{u}'_j\frac{\partial \bar{p}'}{\partial x_j}\right\rangle = \frac{\partial}{\partial x_j}\left\langle \bar{u}'_j\bar{p}'\right\rangle - \left\langle \bar{p}'\frac{\partial \bar{u}'_j}{\partial x_j}\right\rangle \tag{12}$$

where the last term is zero due to continuity.

The last term in Eq. 11 can be both positive and negative. However, if we introduce an eddy-viscosity model it can be shown that it is predominantly negative. If the approximation (using Eq. 10)

$$\tau_{ij}'^{a} = \tau_{ij}^{a} - \langle \tau_{ij}^{a} \rangle = -2\left(\nu_T \bar{s}_{ij} - \langle \nu_T \bar{s}_{ij} \rangle\right) \simeq -2\nu_T s_{ij}' \tag{13}$$

is made we find that the term is always negative. This is easily seen when inserting Eq. 13 into the last term of Eq. 11

$$\left\langle \tau_{ij}'^{a} \frac{\partial \bar{u}_i'}{\partial x_j} \right\rangle \simeq -2\left\langle \nu_T s_{ij}'(s_{ij}' + \omega_{ij}')\right\rangle = -2\left\langle \nu_T s_{ij}' s_{ij}' \right\rangle < 0 \tag{14}$$

where $\omega_{ij}' = 0.5(\partial \bar{u}_i'/dx_j - \partial \bar{u}_j'/dx_i)$. In Eq. 14 we have used the fact that the product of a symmetric and anti-symmetric tensor is zero.

The terms in Eq. 11 have the following physical meaning. The term on the left-hand side is the advection. The terms on the right-hand side are production of $\langle \bar{k} \rangle$, transport of $\langle \bar{k} \rangle$ by resolved fluctuations, viscous transport of $\langle \bar{k} \rangle$, viscous dissipation of $\langle \bar{k} \rangle$ (i.e ε_{sgs}), production/destruction of $\langle \bar{k} \rangle$ by buoyancy, transport of $\langle \bar{k} \rangle$ by SGS/URANS turbulence and production/destruction of $\langle \bar{k} \rangle$ by SGS/URANS turbulence.

To compute ε_{sgs}, we cannot use Eq. 14 cannot use in a CFD code, since we cannot compute a fluctuation without knowing the mean. We *could* run the CFD code twice, computing the mean the first time. However, a better option is

$$\begin{aligned}
\varepsilon_{sgs} &= \left\langle \tau_{ij}'^{a} \frac{\partial \bar{u}_i'}{\partial x_j} \right\rangle = \left\langle \tau_{ij}'^{a}(\bar{s}_{ij}' + \omega_{ij}') \right\rangle = \left\langle \tau_{ij}'^{a} \bar{s}_{ij}' \right\rangle \\
&= \left\langle (\tau_{ij}^{a} - \langle \tau_{ij}^{a} \rangle)(\bar{s}_{ij} - \langle \bar{s}_{ij} \rangle) \right\rangle \\
&= \langle \tau_{ij}^{a} \bar{s}_{ij} \rangle - \left\langle \langle \tau_{ij}^{a} \rangle \bar{s}_{ij} \right\rangle - \left\langle \bar{s}_{ij} \langle \tau_{ij}^{a} \rangle \right\rangle + \langle \tau_{ij}^{a} \rangle \langle \bar{s}_{ij} \rangle \\
&= \langle \tau_{ij}^{a} \bar{s}_{ij} \rangle - \langle \bar{s}_{ij} \rangle \langle \tau_{ij}^{a} \rangle
\end{aligned} \tag{15}$$

1.2 Time-averaged kinetic energy $\langle \bar{K} \rangle$

The equation for the time-averaged kinetic energy $\langle \bar{K} \rangle = \frac{1}{2} U_i U_i$ is derived by multiplying the time-averaged (ensemble-averaged) momentum equation, Eq. 4, by U_i so that

$$U_i \frac{\partial}{\partial x_j}(U_i U_j) = -\frac{1}{\rho} U_i \frac{\partial P}{\partial x_i} + \nu U_i \frac{\partial^2 U_i}{\partial x_j \partial x_j} - U_i \frac{\partial}{d x_j}\left(\langle \tau_{ij}^{a} \rangle + \langle \bar{u}_i' \bar{u}_j' \rangle\right) - U_i g_i \beta T \tag{16}$$

The left-hand side of Eq. 16 can be rewritten as

$$\begin{aligned}
\frac{\partial}{\partial x_j}(U_i U_i U_j) - U_i U_j \frac{\partial U_i}{\partial x_j} &= U_j \frac{\partial}{\partial x_j}(U_i U_i) - \frac{1}{2} U_j \frac{\partial}{\partial x_j}(U_i U_i) \\
&= \frac{1}{2} U_j \frac{\partial}{\partial x_j}(U_i U_i) = \frac{\partial}{\partial x_j}(U_j \langle \bar{K} \rangle)
\end{aligned} \tag{17}$$

The viscous diffusion term in Eq. 16 is rewritten in the same way as the viscous term in Eq. 7, i.e.

$$\nu U_i \frac{\partial^2 U_i}{\partial x_j \partial x_j} = \nu \frac{\partial^2 \langle \bar{K} \rangle}{\partial x_j \partial x_j} - \nu \frac{\partial U_i}{\partial x_j} \frac{\partial U_i}{\partial x_j} \tag{18}$$

The turbulent diffusion term is rewritten as

$$U_i \frac{\partial}{dx_j} \left(\langle \tau_{ij}^a \rangle + \langle \bar{u}_i' \bar{u}_j' \rangle \right) = \frac{\partial}{dx_j} \left[U_i \left(\langle \tau_{ij}^a \rangle + \langle \bar{u}_i' \bar{u}_j' \rangle \right) \right] - \left(\langle \tau_{ij}^a \rangle + \langle \bar{u}_i' \bar{u}_j' \rangle \right) \frac{\partial U_i}{dx_j} \tag{19}$$

Now we can assemble the transport equation for $\langle \bar{K} \rangle$ by inserting Eqs. 17, 18 and Eq. 19 into Eq. 16

$$\begin{aligned} \frac{\partial}{\partial x_j} (U_j \langle \bar{K} \rangle) &= \nu \frac{\partial^2 \langle \bar{K} \rangle}{\partial x_j \partial x_j} - \frac{\partial}{dx_j} \left[U_i \left(\langle \tau_{ij}^a \rangle + \langle \bar{u}_i' \bar{u}_j' \rangle \right) \right] \\ -\frac{U_i}{\rho} \frac{\partial P}{\partial x_i} &+ \left(\langle \tau_{ij}^a \rangle + \langle \bar{u}_i' \bar{u}_j' \rangle \right) \frac{\partial U_i}{dx_j} - \nu \frac{\partial U_i}{\partial x_j} \frac{\partial U_i}{\partial x_j} - g_i \beta U_i T \end{aligned} \tag{20}$$

We recognize the usual transport term on the left-hand side due to advection. On the right-hand side we have the main source term (velocity times the pressure gradient) viscous diffusion and transport of $\langle \bar{K} \rangle$. The term in square brackets represents transport by interaction between the time-averaged (ensemble-averaged) velocity field and turbulence. The term $\langle \bar{u}_i' \bar{u}_j' \rangle \partial U_i / dx_j$ is the usual production term of the resolved kinetic energy $0.5 \langle \bar{u}_i' \bar{u}_i' \rangle$ which usually is negative. This term appears in Eq. 11 but with opposite sign. The term $\langle \tau_{ij}^a \rangle \partial U_i / dx_j$ is the production term in the turbulent kinetic energy equation $k_T = 0.5 \tau_{ii}$. This term is usually referred to as the SGS/URANS dissipation term, and for an eddy-viscosity model we find (cf. Eqs. 13 and 14)

$$\begin{aligned} \langle \tau_{ij}^a \rangle \frac{\partial U_i}{dx_j} &= -2 \langle \nu_T \bar{s}_{ij} \rangle \left(S_{ij} + \Omega_{ij} \right) \\ &\simeq -2 \langle \nu_T \rangle \langle \bar{s}_{ij} \rangle S_{ij} = -2 \langle \nu_T \rangle S_{ij} S_{ij} < 0 \end{aligned} \tag{21}$$

It is interesting to compare this SGS dissipation term with the viscous dissipation term in Eq. 19. If $\nu_{sgs} \gg \nu$, the SGS dissipation is much larger than the viscous one. If this is not the case, then we're doing a DNS!

1.3 Resolved kinetic energy $\bar{K}_{res}$

The equation for the kinetic energy $\bar{K}_{res} = \frac{1}{2} \bar{u}_i \bar{u}_i$ is derived by multiplying the filtered momentum equation, Eq. 1, by $\bar{u}_i$ so that

$$\bar{u}_i \left(\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_k} (\bar{u}_i \bar{u}_k) \right) = \bar{u}_i \left(-\frac{1}{\rho} \frac{\partial \bar{p}}{\partial x_i} + \nu \frac{\partial^2 \bar{u}_i}{\partial x_k \partial x_k} - \frac{\partial \tau_{ik}^a}{\partial x_k} - g_i \beta \bar{t} \right) \tag{22}$$

Looking at the derivation in Section 1.1 and the final equation (Eq. 20) we get

$$\begin{aligned}\frac{\partial \bar{K}_{res}}{\partial t} + \frac{\partial}{\partial x_j}(\bar{u}_j \bar{K}_{res}) = \nu \frac{\partial^2 \bar{K}_{res}}{\partial x_j \partial x_j} - \frac{\bar{u}_j}{\rho}\frac{\partial \bar{p}}{\partial x_j} - \frac{\partial \bar{u}_i \tau^a_{ij}}{\partial x_j} \\ + \tau^a_{ij}\frac{\partial \bar{u}_i}{d x_j} - \nu \frac{\partial \bar{u}_i}{\partial x_j}\frac{\partial \bar{u}_i}{\partial x_j} - g_i \beta \bar{u}_i \bar{t}\end{aligned} \tag{23}$$

1.4 Equation for $K = u_i u_i/2$

The equation for K is derived by multiplying Navier-Stokes (i.e. Eq. 1 without SGS stresses and non-filtered variables) by u_i, i.e.

$$u_i\left(\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_k}(u_i u_k)\right) = u_i\left(-\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_k \partial x_k} - g_i \beta t\right) \tag{24}$$

Looking at the derivation in Section 1.1 and the final equation (Eq. 20) we get

$$\frac{\partial K}{\partial t} + \frac{\partial}{\partial x_j}(u_j K) = \nu \frac{\partial^2 K}{\partial x_j \partial x_j} - \frac{\partial}{\partial x_j}(u_j p) - \nu \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j} - g_i \beta u_i t \tag{25}$$

This is the same equation as in Section 2.3 in [1] but there it is expressed in the stress tensor, σ_{ij}.

1.5 SGS turbulent kinetic energy, $k_T = 0.5(\overline{u_i u_i} - \bar{u}_i \bar{u}_i)$

The SGS turbulent kinetic energy is defined as

$$k_T = 0.5(\overline{u_i u_i} - \bar{u}_i \bar{u}_i) = \bar{K} - \bar{K}_{res} \tag{26}$$

It is obtained by subtracting Eq. 23 from the filtered Eq. 25

$$\begin{aligned}\frac{\partial (\bar{K} - \bar{K}_{res})}{\partial t} + \frac{\partial}{\partial x_j}(\overline{u_j K} - \bar{u}_j \bar{K}_{res}) = \nu \frac{\partial^2 (\bar{K} - \bar{K}_{res})}{\partial x_j \partial x_j} \\ - \frac{\partial}{\partial x_j}(\overline{u_j p} - \bar{u}_j \bar{p} - \bar{u}_i \tau^a_{ij}) \\ - \tau^a_{ij}\frac{\partial \bar{u}_i}{d x_j} - \nu \left(\overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}} - \frac{\partial \bar{u}_i}{\partial x_j}\frac{\partial \bar{u}_i}{\partial x_j}\right) - g_i \beta (\overline{u_i t} - \bar{u}_i \bar{t})\end{aligned}$$

Adding the term $\partial/\partial x_j(\bar{u}_j \bar{K} - \overline{u_j K})$ on both sides and using Eq. 26 gives

$$\begin{aligned}\frac{\partial k_T}{\partial t} + \frac{\partial}{\partial x_j}(\bar{u}_j k_T) = \nu \frac{\partial^2 k_T}{\partial x_j \partial x_j} \\ - \frac{\partial}{\partial x_j}\left(\overline{u_j p} - \bar{u}_j \bar{p} - \bar{u}_i \tau^a_{ij} + \overline{u_j K} - \bar{u}_j \bar{K}\right) \\ - \tau^a_{ij}\frac{\partial \bar{u}_i}{d x_j} \\ - \nu \left(\overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}} - \frac{\partial \bar{u}_i}{\partial x_j}\frac{\partial \bar{u}_i}{\partial x_j}\right) \\ - g_i(\beta \overline{u_i t} - \bar{u}_i \bar{t})\end{aligned}$$

Line 1: convection and viscous diffusion.

Line 2: turbulent diffusion.

Line 3: production; it appears with opposite sign in Eq. 23.

Line 4: viscous dissipation.

Line 5: buoyancy.

1.6 Equation for modeled k_T

The equation for the modelled turbulent SGS/RANS kinetic energy reads

$$\frac{\partial k_T}{\partial t} + \frac{\partial}{\partial x_j}(\bar{u}_j k_T) = \frac{\partial}{\partial x_j}\left[(\nu + \nu_T)\frac{\partial k_T}{\partial x_j}\right] + 2\nu_T \bar{s}_{ij}\bar{s}_{ij} - \varepsilon \tag{27}$$

The terms on the right-hand side represent viscous and turbulent diffusion, production and viscous dissipation.

1.7 Equation for resolved heat flux, $\langle \bar{u}_i' \bar{t}' \rangle$

The filtered temperature equation for $\bar{t}$ reads

$$\begin{aligned} \frac{\partial \bar{t}}{\partial t} + \frac{\partial}{\partial x_k}(\bar{u}_k \bar{t}) &= \frac{\nu}{Pr}\frac{\partial^2 \bar{t}}{\partial x_k \partial x_k} - \frac{\partial h_k}{\partial x_k} \\ h_k &= \overline{u_k t} - \bar{u}_k \bar{t} \end{aligned} \tag{28}$$

Use Eq. 2 in Eq. 28 so that

$$\frac{\partial}{\partial t}(T + \bar{t}_i') + \frac{\partial}{\partial x_k}((U_k + \bar{u}_k')(T + \bar{t}')) = \frac{\nu}{Pr}\frac{\partial^2 (T + \bar{t}')}{\partial x_k \partial x_k} - \frac{\partial h_k}{\partial x_k} \tag{29}$$

Time (ensemble) averaging of Eq. 29 yields

$$\frac{\partial}{\partial x_k}(U_k T) = \frac{\nu}{Pr}\frac{\partial^2 T}{\partial x_k \partial x_k} - \frac{\partial}{\partial x_k}\left(\langle \bar{u}_k' \bar{t}' \rangle + \langle h_k \rangle\right) \tag{30}$$

Now subtract Eq. 30 from Eq. 29

$$\begin{aligned} \frac{\partial \bar{t}'}{\partial t} + \frac{\partial}{\partial x_k}(\bar{u}_k' T + U_k \bar{t}' + \bar{u}_k' \bar{t}') &= \\ \frac{\nu}{Pr}\frac{\partial^2 \bar{t}'}{\partial x_k \partial x_k} + \frac{\partial}{\partial x_k}\left(\langle \bar{u}_k' \bar{t}' \rangle + \underbrace{\langle h \rangle_k - h_k}_{-h_k'}\right) \end{aligned} \tag{31}$$

Multiply Eq. 31 with $\bar{u}_i'$ and multiply Eq. 5 with $\bar{t}'$, add them together and time (ensemble) average

$$\begin{aligned} &\left\langle \bar{u}_i' \frac{\partial}{\partial x_k}(\bar{u}_k' T + U_k \bar{t}' + \bar{u}_k' \bar{t}') + \bar{t}' \frac{\partial}{\partial x_k}(U_i \bar{u}_k' + U_k \bar{u}_i' + \bar{u}_i' \bar{u}_k') \right\rangle \\ &= -\left\langle \frac{\bar{t}'}{\rho}\frac{\partial \bar{p}'}{\partial x_i} \right\rangle + \frac{\nu}{Pr}\left\langle \bar{u}_i' \frac{\partial^2 \bar{t}'}{\partial x_k \partial x_k} \right\rangle + \nu \left\langle \bar{t}' \frac{\partial^2 \bar{u}_i'}{\partial x_k \partial x_k} \right\rangle - g_i \beta \langle \bar{t}' \bar{t}' \rangle \\ &\qquad - \left\langle \bar{u}_i' \frac{\partial h_k'}{\partial x_k} \right\rangle - \left\langle \bar{t}' \frac{\partial \tau_{ik}'^a}{\partial x_k} \right\rangle \end{aligned} \tag{32}$$

The two first lines correspond to the conventional heat flux equation. The two terms in the middle on line 2 can be re-written as

$$\frac{\nu}{Pr}\frac{\partial}{\partial x_k}\left\langle \bar{u}_i'\frac{\partial \bar{t}'}{\partial x_k}\right\rangle - \frac{\nu}{Pr}\left\langle \frac{\partial \bar{u}_i'}{\partial x_k}\frac{\partial \bar{t}'}{\partial x_k}\right\rangle + \nu\frac{\partial}{\partial x_k}\left\langle \bar{t}'\frac{\partial \bar{u}_i'}{\partial x_k}\right\rangle - \nu\left\langle \frac{\partial \bar{u}_i'}{\partial x_k}\frac{\partial \bar{t}'}{\partial x_k}\right\rangle =$$
$$\frac{\nu}{Pr}\frac{\partial}{\partial x_k}\left\langle \bar{u}_i'\frac{\partial \bar{t}'}{\partial x_k}\right\rangle + \nu\frac{\partial}{\partial x_k}\left\langle \bar{t}'\frac{\partial \bar{u}_i'}{\partial x_k}\right\rangle - \left(\nu + \frac{\nu}{Pr}\right)\left\langle \frac{\partial \bar{u}_i'}{\partial x_k}\frac{\partial \bar{t}'}{\partial x_k}\right\rangle \tag{33}$$

Using Eq. 33 in Eq. 32 and at the same time re-writing the SGS/URANS terms we get

$$\begin{aligned}
&\frac{\partial}{\partial x_k}U_k\langle \bar{u}_i'\bar{t}'\rangle \\
&= -\left\langle \bar{u}_i'\bar{u}_k'\right\rangle\frac{\partial T}{\partial x_k} - \left\langle \bar{u}_k'\bar{t}'\right\rangle\frac{\partial U_i}{\partial x_k} - \left\langle \frac{\bar{t}'}{\rho}\frac{\partial \bar{p}'}{\partial x_i}\right\rangle - \frac{\partial}{\partial x_k}\left\langle \bar{u}_k'\bar{u}_i'\bar{t}'\right\rangle \\
&+\frac{\nu}{Pr}\frac{\partial}{\partial x_k}\left\langle \bar{u}_i'\frac{\partial \bar{t}'}{\partial x_k}\right\rangle + \nu\frac{\partial}{\partial x_k}\left\langle \bar{t}'\frac{\partial \bar{u}_i'}{\partial x_k}\right\rangle - \left(\nu + \frac{\nu}{Pr}\right)\left\langle \frac{\partial \bar{u}_i'}{\partial x_k}\frac{\partial \bar{t}'}{\partial x_k}\right\rangle - g_i\beta\left\langle t'^2\right\rangle \\
&-\frac{\partial}{\partial x_k}\left\langle \bar{u}_i'h_k'\right\rangle + \left\langle h_k'\frac{\partial \bar{u}_i'}{\partial x_k}\right\rangle - \frac{\partial}{\partial x_k}\left\langle \bar{t}'\tau_{ik}'^a\right\rangle + \left\langle \tau_{ik}'^a\frac{\partial \bar{t}'}{\partial x_k}\right\rangle
\end{aligned} \tag{34}$$

The SGS/URANS heat fluxes are commonly obtain from an eddy-viscosity model

$$h_i = -\frac{\nu_T}{Pr_T}\frac{\partial \bar{t}}{\partial x_i} \tag{35}$$

1.8 Equation for resolved temperature variance, $\langle \bar{t}'^2\rangle$

Multiply Eq. 31 with $\bar{t}'$ and time (ensemble) average

$$\left\langle \bar{t}'\frac{\partial}{\partial x_k}(\bar{u}_k'T + U_k\bar{t}' + \bar{u}_k'\bar{t}')\right\rangle = \frac{\nu}{Pr}\left\langle \bar{t}'\frac{\partial^2 \bar{t}'}{\partial x_k\partial x_k}\right\rangle - \left\langle \bar{t}'\frac{\partial h_k'}{\partial x_k}\right\rangle \tag{36}$$

The first term on the right-hand side can be re-written as

$$\frac{\nu}{Pr}\left\langle \frac{\partial}{\partial x_k}\left(\bar{t}'\frac{\partial \bar{t}'}{\partial x_k}\right)\right\rangle - \frac{\nu}{Pr}\left\langle \frac{\partial \bar{t}'}{\partial x_k}\frac{\partial \bar{t}'}{\partial x_k}\right\rangle = \frac{1}{2}\frac{\nu}{Pr}\frac{\partial^2}{\partial x_k\partial x_k}\langle \bar{t}'^2\rangle - \frac{\nu}{Pr}\left\langle \frac{\partial \bar{t}'}{\partial x_k}\frac{\partial \bar{t}'}{\partial x_k}\right\rangle \tag{37}$$

Using Eq. 37 and re-writing the SGS/URANS term, Eq. 36 can now be written as

$$\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial x_k}(U_k\langle \bar{t}'^2\rangle) &= -\left\langle \bar{u}_k'\bar{t}'\right\rangle\frac{\partial T}{\partial x_k} - \frac{1}{2}\frac{\partial}{\partial x_k}\left\langle \bar{u}_k'\bar{t}'^2\right\rangle \\
+\frac{1}{2}\frac{\nu}{Pr}\frac{\partial^2\langle \bar{t}'^2\rangle}{\partial x_k\partial x_k} - \frac{\nu}{Pr}\left\langle \frac{\partial \bar{t}'}{\partial x_k}\frac{\partial \bar{t}'}{\partial x_k}\right\rangle &- \frac{\partial}{\partial x_k}\left\langle \bar{t}'h_k'\right\rangle + \left\langle h_k'\frac{\partial \bar{t}'}{\partial x_k}\right\rangle
\end{aligned} \tag{38}$$

Multiply Eq. 36 by 2 and we get

$$\begin{aligned}\frac{\partial}{\partial x_k}(U_k\langle \bar{t}'^2\rangle) = -2\left\langle \bar{u}'_k\bar{t}'\right\rangle\frac{\partial T}{\partial x_k} - \frac{\partial}{\partial x_k}\left\langle \bar{u}'_k\bar{t}'^2\right\rangle \\ +\frac{\nu}{Pr}\frac{\partial^2\langle \bar{t}'^2\rangle}{\partial x_k\partial x_k} - 2\frac{\nu}{Pr}\left\langle\frac{\partial \bar{t}'}{\partial x_k}\frac{\partial \bar{t}'}{\partial x_k}\right\rangle - 2\frac{\partial}{\partial x_k}\left\langle \bar{t}'h'_k\right\rangle + 2\left\langle h'_k\frac{\partial \bar{t}'}{\partial x_k}\right\rangle\end{aligned} \tag{39}$$

References

[1] L. Davidson. Fluid mechanics, turbulent flow and turbulence modeling. eBook, Division of Fluid Dynamics, Dept. of Mechanics and Maritime Sciences, Chalmers University of Technology, Gothenburg, 2021.