Transition Modelling – A Review

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Abstract

This review article discusses the phenomenon of laminar to turbulence transition in general and transition due to elevated freestream disturbances in particular. The motivation of the study is primarily the increased demands from the gas turbine industry of accurate heat transfer predictions in high and low pressure turbines. In these devices the trends of the last decade, e.g. the development of smaller turbines and turbines with higher by-pass ratios (to reduce noise), lead to engines operating at lower Reynolds number conditions and, thus, delayed and larger transitional flow regions.

Three different modes of transition are discussed–natural, by-pass and separation induced transition. Natural transition is (fortunately) not an issue in gas turbine design, nor suitable to statistical modelling, which indeed is the closure level the authors intend to adopt, and is therefore not dealt with in detail. By-pass and separation induced transition on the other hand are probably about equally important to gas turbine flows. The body of literature dedicated to the latter is, however, not as considerable as that dealing with by-pass transition. This is one of the reasons why this article sometimes is biased towards by-pass transition modelling. The other reason is that this is the transition mode best suited to be described with a statistical approach, which has tempted many authors to apply well established ideas from the vast field of turbulence modelling.

After an introductory description of the different transition modes some early work on transition is discussed. The turbulent spot concept (Emmons, 1951), followed by a derivation of the Dhawan & Narasimha (1958) intermittency distribution, is of coarse included. With the introduction of intermittency, there was soon a need for a theoretical analogue to the experimental technique 'conditional sampling', i.e. a measure to treat, or model, turbulent and laminar flow portions separately. Conditional averages were

introduced by Libby (1974) and it was not long before full transport equations for the intermittency factor was derived. Discussed is also the prospect of using so called low Reynolds number modelling of *both* near wall turbulence and transition. Thereafter follows a description of two recent transition modelling approaches and the review ends with a section collecting numerous modelling attempts, ranging from simple algebraic correlations to complex transport equations for the intermittency factor, all aiming to improve predictions of the intriguing physics of transitional flows.

Nomenclature

Latin Symbols

C_p, C_f	Pressure and friction coefficients
f	Relaxation variable
g	Spot production rate per unit area
I	Indicator function
K	Acceleration parameter, $K \equiv \nu U^2 \frac{dU}{ds}$, cf. Eqn 92-93
k	Kinetic energy, $k \equiv \frac{1}{2}\overline{u_i u_i}$
k_L	Laminar (nonturbulent) kinetic energy
k_T	Turbulence kinetic energy
L	Turbulence length scale
P	Mean pressure
p'	Fluctuating pressure
P_k	Turbulence production term
S_{ij}	Mean rate of strain tensor
T	Turbulence time scale
Tu	Turbulence intensity, $Tu \equiv \frac{\left(\frac{1}{3}\overline{u_i u_i}\right)^{1/2}}{U}$
U_i	Mean velocities in x_i-direction
u_i'	Velocity fluctuations
v^2	Turbulence velocity scalar
$\overline{u_i u_j}$	Reynolds stresses
W_{ij}	Mean rate of strain tensor
$\dot{w}$	Volumetric creation of I

Greek Symbols

ε	Dissipation rate of k
γ	Intermittency factor
ν	Kinematic viscosity
ν_t	Eddy-viscosity
ϕ	Flow variable
ρ	Density
λ_θ	Pressure gradient parameter, $\lambda_\theta \equiv \frac{\rho\theta^2}{\mu}\frac{dU}{ds}$
σ	Spot propagation parameter
θ	Boundary layer momentum thickness
τ_w	Wall friction

Superscripts and/or subscripts

$+$	Variable normalized using wall parameters
L	Conditioned laminar zone value
T	Conditioned turbulent zone value
w	Wall value
∞	Freestream value

Averages

$\langle \phi \rangle$	Ensamble average of ϕ
$\overline{\phi}$	Time average of ϕ

1 Introduction

With the last decades of intense research in turbulence modelling statistical descriptions of turbulence are about to mature. Today closures exist that are able to provide accurate predictions of turbulence effects on the mean flow characteristics given that the mean flow itself is not too complicated. In flows where more complex flow features are present turbulence models still lack in reliability and require a more careful (critical) analysis of computed data, at least until new even more powerful methods are developed.

However, there is one phenomenon that can cause the most advanced turbulence model to fail in the, seemingly, most straightforward flows to compute, e.g flow over a flat plate. This phenomenon is the transition of a laminar flow into a turbulent state. Transition has been studied extensively since Osbourne Reynolds, in 1883, was able to relate the parameter $\rho V d/\mu$ to the change in flow behavior as the flow transitions to turbulence (note that there had been reports on this peculiar phenomenon as early as 1839). The reason for the great interest in transition is not only that it plays an important role in many engineering applications but also that it raises a more fundamental question on the nature of flow physics and is an example of the problem of determinism and chaos. Not until recently, with the aid of Direct Numerical Simulations, the details of the mechanisms behind transition begin to become clearer. Note that DNS, due to the requirement of resolving all scales of motion, is still limited to very simple flows, mostly of academical interest.

A particular field that recently has shown an increased interest in transition physics and its modelling is that of turbomachinery design. In gas turbines for example, transitional phenomena are crucially important to the design of the compressor and the turbines. In the former, about half the loss of stage efficiency at the design point owes to skin friction, which is several times larger in a turbulent boundary layer than in its laminar counterpart. When the compressor is run at off design conditions, however, losses commonly rise rather dramatically as an effect of fbw separation. The extent of the separated regions is influenced by the state of the fbw in the separated shear layer. If the separation bubble, which usually is laminar at the point of separation, transitions it will likely reattach to the blade surface and the loss in efficiency is limited. Under some circumstances, the transitional process is slow and the fbw reattachment point may move far downstream on the suction surface and the large separated region cause severe losses is stage efficiency. It thus becomes evident that there is a trade off between the increase in skin friction losses at design and the risk of massive separation at off design. The compressor designer must also be aware of the complex interplay between separation and transition at off design conditions. These arguments also apply to the turbine of the gas turbine engine. Here the high temperature environment further complicates the situation and boundary layer transition (to turbulence) dramatically increases the heat load on the turbine airfoils.

Today, several trends lead to gas turbines operating at reduced Reynolds numbers. One is that the market of small turbines (25-100 kW) is increasing (UAVs powered by gas turbines, intracity Internet links, military vehicles). Another is the trend towards higher by-pass ratio engines with reduced noise levels. Lower Reynolds numbers lead to boundary layers that are less likely to transition to a turbulent state, and when they do, the process is usually slower and the extent of transitional boundary layers along the blade surfaces increases. Consequently, the need for accurate predictions of transitional boundary layers also increases.

This literature survey aims at providing short descriptions of three different ‘modes’ of transition – natural, by-pass and separation induced transition. Then follows a section that provides a historical perspective in which several transition related concepts, such as spot propagation rate and intermittency, are introduced together with a few different modelling approaches. Finally, some more recent approaches to transition modelling are discussed. Note that this study was performed to find a basis on which a suitable transition modelling approach could be coupled to the $v^2 - f$ model to improve that model's ability to predict, primarily, by-pass transition. Therefore, this report is sometimes biased towards by-pass related research. This limitation is however not that serious as natural transition is probably beyond the scope of modelling and that modelling of separation induced transition is not as well developed as by-pass transition modelling.

2 Transition modes

2.1 Natural transition

Natural transition refers to a breakdown process of a laminar boundary layer that contains several types of growing instabilities that eventually cause the boundary layer to transition to being turbulent. Initially the fluid responds to some weak disturbance, like surface waviness or vibration, causing a small deviation from the laminar base flow. These disturbances then lead to what usually is referred to as primary instabilities, e.g. Tollmien-Schlichting waves in boundary layers. The primary instabilities are linear, usually viscous and grow slowly until they reach an amplitude sufficiently large to trigger instabilities of streamwise periodic nature – so called secondary instabilities. These instabilities are unstable and grow rapidly into three-dimensional structures that very soon breaks up to form isolated spots of turbulent flow. Finally these spots grow and eventually coalesque into a fully turbulent boundary layer. As natural tranistion is of minor importance in turbomachinery applications this review will not discuss the subject any further. The research on fundamental aspects of natural transition and stability led by Prof. Henningson at the Royal Institute of Technology, in Stockholm, Sweden is among the most acknowledged and for further information on the subject the reader is referred to Schmid & Henningson (2000).

2.2 By-pass transition

As mentioned above the occurence of transition requires some kind of initial disturbance. When this disturbance is weak the boundary layer will take its 'natural' route to turbulence. However, in many applications, e.g. at the exit of a gas turbine burner, the disturbances are far from weak and the obvious question is whether or not the elevated amplitude of the disturbances have an effect on the transition process. Indeed they do. It turns out that when a laminar boundary layer is subjected to disturbance levels above a certain amplitude some of the initial stages in the natural transition process never take place. Instead these, by nature slowly evolving, instabilities are 'by-passed' by some presumably non-linear, unsteady modes that accelerate the breakdown of the boundary layer into turbulence spots.

Traditionally most of the research has focused on the events of natural transition. Here the primary (small) instabilities can be described with linear theory, to which analytical solutions can be found, that provides a solid base for the analysis of the dynamics of the instabilities. In by-pass transition on the other hand it becomes essential to capture the coupling between large freestream disturbances and the pretransitional boundary layer in order to correctly describe transtion. This task is difficult and requires, in principle, DNS, which is the reason why the underlying physics of by-pass transition is not yet very well understood.

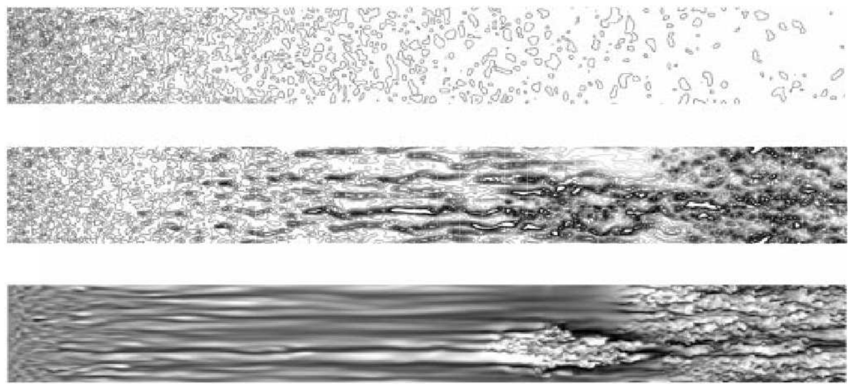


Figure 1: Instantaneous snapshot of u velocity contours at three hights above a transitional flat plate flow as computed by Jacobs & Durbin (2001). Reprinted by permission of Cambridge University Press.

Simon & Kaszeta (2001) reviewed the work done to characterize and model by-pass transition (and wake-induced separation which was regarded as a subset to by-pass transition). They stress that the trend towards turbine designs with lower Reynolds numbers and increased blade loadings makes the role of transition as a separation preventing tool increasingly important.

In an effort to provide the data necessary to depict the interaction of freestream turbulence with a laminar boundary layer Jacobs & Durbin (2001) performed a DNS of a flate plate flow in an environment of elevated turbulence intensity, $Tu = 3.0\%$. Their results suggest that low frequency perturbations in the freestream are able to penetrate the boundary layer, where they produce boundary layer modes of even lower frequency. These modes in turn, are acted upon by shear and grow and elongate in the direction of the flow. The growth of these streaks is essentially due to vertical displacement of mean momentum (Jacobs & Durbin, 2001).

The streaks are visualized in Figures 1 and 2. In Fig. 1 contours of the streamwise velocity component are shown at three heights above the wall. At the top plane, located in the freestream, the velocity fluctuations are isotropic. In the middle plane, at the edge of the boundary layer, there is a change of pattern and the above-mentioned elongated streaks become visible. Further into the boundary layer (bottom plane) the streaks are even clearer (as the small scales eddies in the freestreem do not penetrate this deep) and the larger streak in the bottom of the view is seen to be a precursor to a turbulent spot. Figure 2 illustrates that low frequency structures only exist in the u velocity component. Hardly any perturbations in the v velocity is seen upstream of the creation of the turbulent spot in which standard turbulence mechanisms redistribute fluctuating energy amongst the three velocity components.

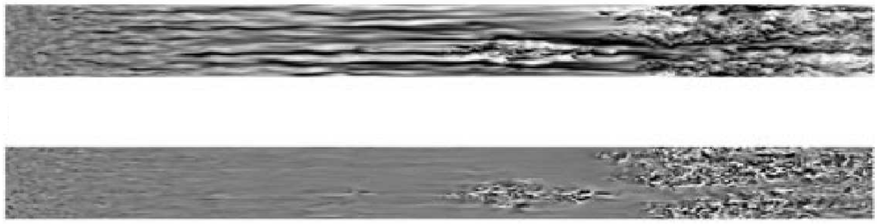


Figure 2: Instantaneous snapshot of u and v velocity contours as computed by Jacobs & Durbin (2001). Reprinted by permission of Cambridge University Press.

Still, it is not all clear that the streaks in the laminar boundary layer play a significant role in the further development towards transition. It seems that these modes too have difficulties to correlate with freestream disturbances due to a sheltering effect of the shear layer. However, a mechanism that possibly bridges the buffeted boundary layer with the freestream was identifed. It consists of certain streaks in the form of long, intense jets moving backwards relative to the mean motion. These jets can be found throughout the boundary layer and seem to be able to couple with small-scale turbulence when they exist at the very surface of the boundary layer. Here the sheltering effect is overcomed as the jet introduces a three-dimensional, low-frequency flow, which is acted upon by freestream eddies of rather small scales.

Most of the results presented by Jacobs & Durbin (2001) were reproduced and complemented by additional DNS by Brandt *et al.* (2004). The latter authors performed several simulations at different freestream intensities and varied the length scale and spectral composition of the freestream turbulence in an effort to shed further light on the growth mechanisms of the laminar kinetic energy and its influence on the laminar to turbulent breakdown process. They show that the growth of the laminar perturbations under certain freestream conditions (low-frequency freestream perturbations) behaves as

$$u_{rms} \sim Re_x^{1/2} Tu, \tag{1}$$

a growth predicted by Brandt *et al.* (2002) using a nonlinear model for receptivity. In other circumstances (high-frequency freestream perturbations) the perturbations develop as

$$u_{rms} \sim Re_x^{1/2} Tu^2, \tag{2}$$

which can be predicted by a linear ‘lift-up’ mechanism. The streak/turbulent spot breakdown mechanism suggested by Brandt *et al.* (2004) is associated with high-frequency motions of low-speed streaks. These instabilities owe to strong spanwise shear and when they produce regions of positive and negative wall normal and spanwise velocities breakdown is likely to occur. Finally, their simulations confirm a dependence on the freestream turbulence length scale. It is shown that increasing the length scale promotes the onset of transition in accordance with the shear sheltering effect discussed in Jacobs & Durbin (2001).

Of even greater interest to the turbulence modelling community are the *statistical* footprints of freestream disturbances in the pretransitional boundary layer. The LES of Lardeau *et al.* (2005) suggests that the main contributor to the growth of laminar kinetic energy[1] is that of Reynolds stresses acting on a mean flow gradient. This is the same mechanism as in fully turbulent flows with the only difference that the shear to normal stress ratio ($\overline{uv}/\overline{u_i u_i}$) is some 30-50 percent lower. This result is supported by the measurements of Volino & Simon (1994) and Volino & Simon (1997*b*). They found significant levels of $\overline{uv}$ in the boundary layer prior to transition and it occurred at frequencies of the peak v' motion of the freestream unsteadiness. The u' fluctuations also appeared at this frequency. The levels of the correlated motions were not comparable to those of a turbulent boundary layer but large enough to influence the mean velocity and temperature profiles and thus able to cause elevations in skin friction and heat transfer (Volino & Simon, 1997*a*). Volino (1998) suggests that the boundary layer fluctuations are due to the splat mechansim (cf. e.g. Bradshaw, 1996). Finally, the shear sheltering mechanism proposed by Jacobs & Durbin (2001) finds support from measurements of Thole & Bogard (1996) and Moss & Oldfield (1996). Both studies show that high frequency fluctuations have only limited effect on near wall mean flow characteristics.

To illustrate that laminar streaks are present also in highly disturbed gas turbine-like environments consider the measurements of Radomsky (2000). He studied the influence of high freestream turbulence levels (up to $Tu = 19.5\%$) on boundary layer characteristics in a linear cascade of turbine stators. These measurements indicate the presence of laminar streaks, at least on the pressure side of the stator, where they found the u' fluctuation profiles characteristic of the laminar fluctuations. As an example the root mean square values of the streamwise and wall-normal fluctuations, u_{rms} and v_{rms}, respectively, of the $Tu = 19.5\%$ case of Radomsky (2000) are shown in Figure 3. It can be seen that the trends in the u_{rms} and v_{rms} profiles are not the same (v_{rms} monotonically decreasing whereas the peak value of u_{rms} is approximately constant) which indicates the presence, and growth, of laminar streaks. Note that these streaks seem to cover the full pressure side of the vane and that a strong acceleration prevents the boundary layer from transition to turbulence (cf. Radomsky (2000); Radomsky & Thole (2001)).

Another study on freestream turbulence/streamline curvature interaction with relevance to gas turbine design is that of Schultz & Volino (2003). They investigated the boundary layer on a concave wall beneath a disturbed and accelerating freestream and found profiles similar to those seen in Figure 3. Some of their re-

[1]Here, and throughout this report, the term laminar kinetic energy refers to energy of fluctuations that are not turbulence

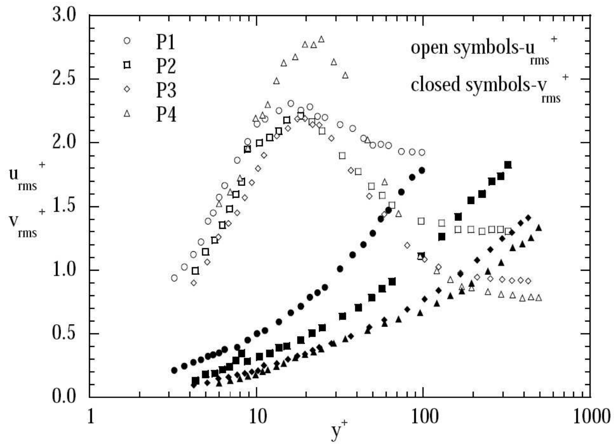


Figure 3: Variation of streamwise (u^+_{rms}) and wall normal (v^+_{rms}) fluctuations across the boundary layer of a stator vane beneath a highly disturbed freestream ($Tu = 19.5\%$ about one chord upstream of the leading edge). The measurement stations P1-P4 are located 0.15, 0.3, 0.45 and 0.6 chord length downstream of the stagnation point (the distance measured along a vane surface coordinate). The Figure is reprinted from Radomsky (2000).

sults (small elevations of near wall u' and v' relative to a flat wall boundary layer) suggested the existence of Görtler vortices but the existence of such vortices could not be confirmed. Note that the boundary layer measured was transitional and that the data was conditioned to separately display features of laminar and turbulent parts of the flow. Finally, one of the most frequently cited experimental studies of by-pass transition is that of Roach & Brierley (1992). They collected the data of the T3A and T3B test cases that have become the test cases against which closure models' transitional behavior are usually first validated.

2.3 Separation induced transition

> Of all the transition modes, there is none more crucial to compressor and low-pressure turbine design and none more neglected than this mode. *Mayle (1991)*

The molecular diffusion in laminar turbomachinery boundary layers is often small as compared with the existing pressure gradients in, for example, a gas turbine compressor. The transport of streamwise momentum across the boundary layer is therefore not sufficiently large to prevent a deceleration of the innermost fluid which eventually begins to flow backwards to form a so called separation bubble. The nature of this bubble is, at least initially, laminar. The bubble/main flow laminar shear layer is however highly unstable with three dimensional instabilities. Just as for the other two modes of transition these instabilities may break down to form turbulent spots where after the shear layer soon becomes fully turbulent. Once transition has occurred the cross-streamwise transport of momentum increases drastically, which often allows the flow to reattach to the surface from which it separated. The new *turbulent* boundary layer, with mixing properties far better than those of the molecular diffusion mechanism in the laminar boundary layer, is now able to withstand far greater pressure gradients without separating again. This type of separation bubble is often referred to as a laminar separation bubble even if it reattaches in a turbulent state. The term transitional separation bubble would seem more appropriate as it helps distinguish between completely laminar or turbulent separation bubbles.

Worth mentioning here is that some separation bubbles only act to displace the above fluid almost without affecting the pressure distribution. These bubbles are referred to as short bubbles and act only as to trigger transition. Long bubbles on the other hand have an affect on the overall flow picture and can lead to significant losses in lift. A problem in gas turbine design is that small changes in flow conditions can transform a small bubble into a long with dramatic consequences on the gas turbine performance. This phenomena is commonly referred to as 'bubble bursting'.

As the focus of this review is on by-pass transition modelling separation induced transition will not be equally well covered. Therefore we will only mention a few selected studies that the present authors believe to give a proper starting point for further reading. Volino & Hultgren (2001) and Volino (2002) experimentally studied separated flow transition under low-pressure gas turbine-like conditions. Among their findings are that the boundary layer begins to reattach shortly after transition has occurred and that the reattachment process is strongly dependent on both Reynolds number and turbulence intensity of the freestream. It is also pointed out that high freestream turbulence levels increase losses by promoting boundary layer growth *but* may also prevent strong separation as transition in the separated shear layer is promoted, which can reduce the overall loss. Another study along the same line is that of Malkiel & Mayle (1996). This article does also include a short review of earlier attempts to model the intriguing physics of transitional separation bubbles.

Häggmark (2000) performed a study of more fundamental nature of the breakdown processes in laminar separation bubbles. Detailed measurements in the bubble/freestream shear layer showed high frequency instability waves growing exponentially leading to a rapid transition process. A review on the same subject was written by Dovgal *et al.* (1994).

2.4 Some Flow Parameters that Influence Transition

2.4.1 Freestream Turbulence

As evident from Section 2.2 both the level (Tu) and nature (length scale, frequency composition, anisotropy etc.) of disturbances in the freestream are of great importance to the laminar to turbulent breakdown process. Its influence is not only important in attached boundary layer transition (by-pass transition) but equally important in separation induced transition, where it many times determines whether or not a laminar separation bubble will transition rapidly enough to prevent bubble bursting.

2.4.2 Pressure Gradients

Favorable pressure gradients, i.e. a pressure decreases in the flow direction, acts as to delay transition. If the favorable pressure gradient is strong enough it may even cause reverse transition–relaminarisation, which, according to Mayle (1991), may happen when the acceleration parameter, $K = \nu U^2(dU/dx)$, reaches levels of approximately 3.2×10^{-6} or higher. Adverse pressure gradients on the other hand promote both transition and separation. As turbulent boundary layers can withstand stronger adverse pressure gradients than laminar boundary layers transition can be used to prevent separation. It is therefore very important to understand the influence of adverse pressure gradients on *both* transition and separation.

2.4.3 Flow Unsteadiness–Passing Wakes

Flow unsteadiness causing transition, sometimes referred to as wake induced transition, will here not be dealt with separately. The somewhat crude assumption is made that the increased levels of turbulence in an oncoming wake will simply act as a region of elevated turbulence. Thus, wake induced transition is regarded as a subset of by-pass transition and may probably be computed with models similar to those used to predict by-pass transition. As an example the effect of passing wakes has with reasonable success been predicted with unsteady RANS $v^2 - f$ turbulence model computations (Wu & Durbin, 2000).

2.4.4 Flow Curvature

The effect of flow curvature, apart from the fact that it causes pressure gradients, is probably not very strong. There is a possibility, on concavely shaped surfaces, that Görtler vortices may exist (Schultz & Volino, 2003) and that they might affect the transitional behavior.

2.4.5 Surface Roughness

The influence of surface roughness in gas turbines is a nontrivial subject. Large local variations in roughness heights exist and when sufficiently large they cause by-pass transition. However, the database compiled by Bons *et al.* (2001) reveals that rms values of roughness heights typical for low-pressure turbine blades lie in the range of 2-9 μm. These levels tend to be too small to cause by-pass transition in flow conditions typical of the blades investigated. Thus, studying the effect of roughness on separation induced transition is probably of greater value. This was done by Roberts & Yaras (2005) who concluded that the effect of surface roughness is comparable to that of freestream turbulence and that the roughness distribution (height, spacing and skewness of roughness elements) is important to consider when quantifying its effect on transition in separation bubbles.

2.4.6 Compressibility

The effects of Mach number on transitional flows are probably rather weak. For example, Boyle & Simon (1998) studied the influence of compressibility and found that only for $Ma > 2$ some modifications to the spot production rate were needed.

2.4.7 Other Disturbances

In some applications, e.g. the high-pressure turbine in a gas turbine engine, it is common to have devises that directly triggers transition. Examples are film and endwall cooling, sudden changes in geometry (e.g. steps or trip wires that causes separation induced transition), imperfections in solid surfaces (again leading to separation) and so on. If these types of strong boundary layer disturbances are known in advance it, in principle, allows the CFD user to trigger transition at a certain, given location and thus, removes the need to predict transition.

3 Modelling Approaches

With the most recent results from state-of-the-art direct simulations presented in Section 2.2 in mind earlier work on transition will now be addressed. The review starts with the classical spot theory of Emmons and how it was extended to yield the algebraic correlations that today are commonly used in gas turbine engine design. Also described is the development of so called conditioned equations that is presently being used in transition research headed by Prof. E. Dick at Delft University. Further, some comments on the application of classical turbulence modelling to transitional flows are given. Finally, this section ends with a few recently proposed approaches to the problem of predicting transitional effects on mean flow quantities. It should also be emphasized that all the methods to be presented are aimed at being incorporated in numerical schemes that treat turbulence in a statistical manner. Therefore, the prospect of using LES or other versions of multi-point closures is excluded. These methods are regarded as still being to expensive to use in everyday industrial applications.

3.1 The intermittency concept

Much of today's research in transition modelling stem from the classical work of Emmons in the early 1950's. He found, contrary to that time's belief of transition being a more or less distinct two-dimensional breakdown of a laminar flow, that transition is a random process which can be pictured by production of isolated turbulent spots in an otherwise laminar environment. After being formed the spots grow as they are convected downstream and will eventually coalesce to form what is referred to as a fully turbulent boundary layer. This scenario is depicted in Fig. 4. Today we know that this description lack some details of the initial stages of natural transition like the formation of Tollmien-Schlichting waves that grow and develop into three-dimensional structures that usually precede the formation of spots. However, these early stages are not relevant in gas turbine engine design (the fan excepted) as they are by-passed due to high levels of freestream disturbances. In this case the production of spots is not triggered by instabilities within the pretransitional boundary layer but rather by, as previously mentioned, certain disturbances of certain frequencies in the freestream that the boundary layer happens to be especially receptive to. Further, as the spot creation process appears to be more or less random, the events described by Emmons are probably to be sophisticated enough to form a basis on which statistical transition modelling in gas turbine engines can be built.

Based on his observations of the nature of transition Emmons defined a spot production rate per unit area, $g(x_0, z_0, t_0)$ (restricted to solid wall surfaces), and derived an expression for the fraction of time the flow over any point (x, z) is turbulent due to the growth/convection of the spots produced at (x_0, z_0, t_0) – the intermittency, $\gamma(x, z)$

$$\gamma(x, z) = 1 - \exp\left[-\int\int\int_R g(x_0, z_0, t_0)\, dx_0\, dx_0\, dt_0\right] \tag{3}$$

where R denotes integration over the so called *dependence volume* (see Figures 5 and 6 that illustrate the propagation of a spot in (x-z-t)-space).

As an example of how to use this equation to determine γ Emmons considered, in lack of experimental data suggesting anything else, a $g(x_0, z_0, t_0)$ distribution over a plate that is independent of both space and time, i.e. $g(x_0, z_0, t_0) = g$. Further, he assumed that all points on the spot boundary propagate at a constant

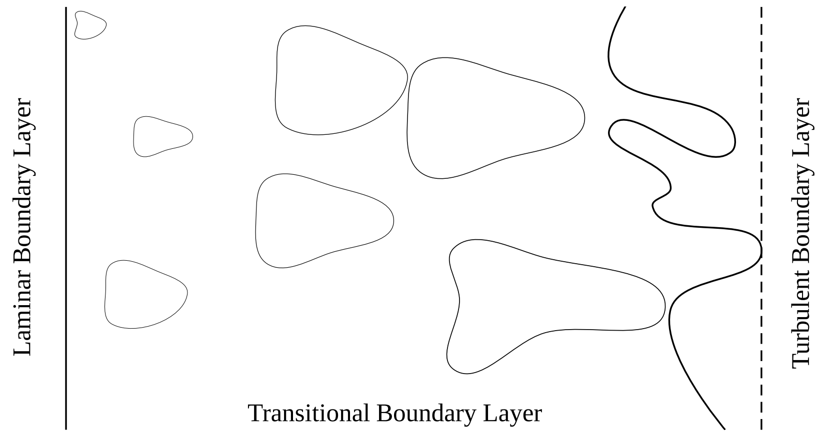


Figure 4: Schematic over instantaneous spot growth in the transitional region. Note that the shape of the spots differ from the one seen in the by-pass DNS (cf. Figures 1 and 2). The ones shown here are characteristic of natural transition.

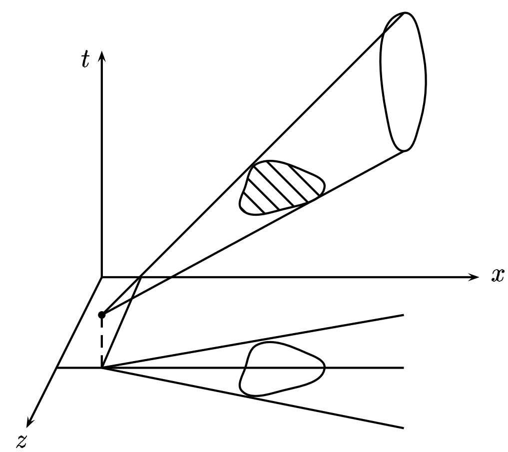


Figure 5: Propagation of a turbulent spot in (x-z-t)-space.

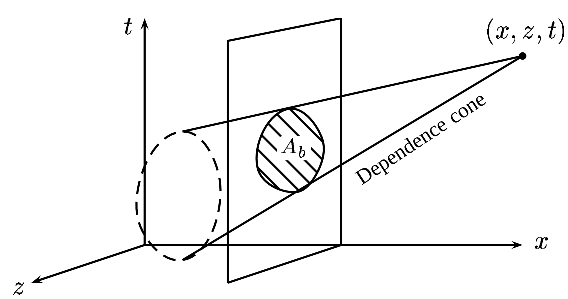


Figure 6: Schematic of dependence volume of the point (x, z, t). All points within the cone may produce a turbulent spot that affects (x, z, t). Also illustrated is the time-width shape (A_b) of the cone at a given x-location.

velocity away from its source. This gives a dependence volume in the form of a true cone with straight generators (Fig. 6). Performing the above integration then simply gives

$$\int_R g\,dV = gV \tag{4}$$

where $V \quad [m^2 s]$ is the volume of the dependence region in $(x\text{-}z\text{-}t)$-space which needs to be determined. As different parts of the spot boundary propagate with different speeds the cone is not circular and computing its volume is therefore not straightforward. Emmons attacked the problem by writing the volume as a base area, A_b, times the height of the cone, x

$$V = \frac{A_b x}{3} = \frac{A_1 x^3}{3} \tag{5}$$

where A_1 is considered to be the "time-width shape" of a spot and was calculated by Emmons as

$$A_1 = \frac{\sigma}{U} \tag{6}$$

where σ is a dimensionless spot propagation parameter governed by geometrical parameters of the spot and spot velocity/freestream velocity ratio. For further details the reader is referred to the original paper of Emmons (1951).

After performing the integration, we arrive at

$$\gamma(x) = 1 - \exp\left[-\frac{\sigma g x^3}{3U}\right] \tag{7}$$

Here Emmons suggested that even though g was assumed to be evenly distributed over the plate it can be taken as $g = 0$ for $x < x_t$, where x_t is some critical location downstream of the plate leading edge. The consequence is that the apex of the domain of dependence is shifted downstream by x_t giving

$$\gamma(x) = 1 - \exp\left[-\frac{\sigma g (x - x_t)^3}{3U}\right] \qquad (x > x_t) \tag{8}$$

Finally, Emmons assumed properties of a transitional boundary layer, say ϕ, can then be obtained from a linear superposition of the laminar and turbulent solutions according to

$$\phi = (1 - \gamma)\phi_L + \gamma\phi_T \tag{9}$$

Transition models that combine a laminar and a turbulent solution in this manner was later (Narasimha & Dey, 1989) referred to as 'linear combination models'. Up to the mid 1970s this was the most commonly used approach to model transitional effects.

One question only briefly touched upon by Emmons is how the spot production rate, $g(x_0, z_0, t_0)$, should be modelled. This subject was addressed by Dhawan

& Narasimha (1958). They made the observation that in flat plate experiments the production of turbulent spots is restricted to a rather narrow region at a finite distance from the leading edge. Therefore they assumed g to be given by a Gaussian error function with its maximum at the transition onset and used different values of the standard deviation in order to match measured intermittency profiles. They found that the lower the deviation the better was the agreement with experimental data. This suggested that the production rate could be approximated using the Dirac delta function according to

$$g(x_0, z_0, t_0) = n\delta(x_0 - x_t) \tag{10}$$

where n is a spot production rate per unit spanwise distance, z, and x_t is a streamwise location from which most (all) turbulent spots are assumed to emanate. When this equation is substituted into Eqn 3 we obtain

$$\begin{aligned} \gamma(x) &= 1 - \exp\left[-\int_R n\delta(x_0 - x_t)\, dx_0\, dz_0\, dt_0\right] \\ &= 1 - \exp\left[-nA_b|_{x-x_t}\right] \end{aligned} \tag{11}$$

where $A_b|_{x-x_t}$ is the cross section area of the dependence domain at the streamwise position $(x - x_t)$. Following the analysis of Emmons, in short described above for the case of an evenly distributed source, this expression can be written as

$$\gamma(x) = 1 - \exp\left[-\frac{n\sigma}{U}(x - x_t)^2\right], \qquad (x \geq x_t) \tag{12}$$

Note that this expression was originally (Dhawan & Narasimha, 1958) written on the slightly different form

$$\begin{aligned} \gamma(x) &= 1 - \exp\left[-A\left(\frac{x - x_t}{\lambda}\right)^2\right] \\ \lambda &= x|_{\gamma=0.75} - x|_{\gamma=0.25} \end{aligned} \tag{13}$$

where A is a constant equal to 0.412. Also note that this equation was not used to predict transition but rather to determine the location of transition onset from experimental data. Now the standard procedure is to rewrite this equation using the local Reynolds number and a dimensionless spot production parameter $\hat{n} = n\nu^2/U^3$

$$\gamma(x) = 1 - \exp\left[-\hat{n}\sigma\,(Re_x - Re_{xt})^2\right], \qquad (Re_x \geq Re_{xt}) \tag{14}$$

This equation is at first sight a straightforward description of a very complex phenomenon and has been used, probably due to its simple formulation, extensively in investigations where transition play an important role. Several researches have also made modifications to the model in order to increase its validity. Recall that the model originally was developed based on observations of a transitional flat plate boundary layer in a zero pressure gradient environment and should not be expected to work equally well for more complex flows in arbitrary geometries. As an example Chen & Thyson (1971) reformulated the model to suite axisymmetric bodies with varying freestream (boundary layer edge) velocity. In later sections other suggested modifications will be discussed in brief.

Unfortunately, Eqn 14 is not closed but requires prescription of the spot production parameters, $\hat{n}$ and σ (or $\hat{n}\sigma$), and also the transition onset location, here represented by the transition onset Reynolds number, Re_{xt}. Providing correlations, valid for a wide range of flow conditions, for these quantities has been a challenging and very active research area ever since the above form of the intermittency function was proposed. Therefore, a vast number of correlations has been suggested and applied to transitional flows with various success. In Section 4.2 a number of these correlations will be presented.

Note that so far only the streamwise variation of the intermittency distribution has been discussed. The reader should be aware that there also exists a wall-normal variation. For a transitional boundary layer in an undisturbed environment γ is relatively large (close to unity) near walls and decreases as the edge of the boundary layer is approached. If the surrounding freestream on the other hand is turbulent (disturbed) γ cannot be expected to vanish at the boundary layer edge but should also here be about unity. Recall that in by-pass transition it is the turbulence in the freestream that triggers turbulence production. This view of intermittency, i.e. that it should have a finite value in a turbulent freestream, is fairly new (suggested by Yang & Shih (1991)) and has been adopted by, for example, Menter *et al.* (2004).

Another class of flows with intermittent behavior needs to be mentioned here. They are flows in which a turbulent 'core' flow entrains irrotational flow from an undisturbed surrounding. As pointed out by Patel & Scheuerer (1982) a turbulence model tuned for fully turbulent flows will probably not work equally well in a region that to some extent consists of non-turbulent flow. Typical examples are turbulent jets in a stagnant surroundings and wake flows.

3.2 Conditioned Equations

In most flows of practical interest there exists regions that can be characterized as being either laminar, fully turbulent or transitional. The transitional region may be thought of as a region with subregions that can be either laminar or turbulent (e.g. turbulent spots in an otherwise laminar environment). Consider for example the turbulent jet that exits in a quiescent surrounding. Clearly, the core of the jet will be turbulent and regions away from the jet laminar. Close to the edge of the jet on the other hand the state of the flow is less obvious. Here the turbulent jet entrains fluid from the laminar region, the two types of flow mixes and we can expect a very complex interface between laminar and turbulent regions. Another example is the flow over a flat plate. Initially the flow is laminar but as the flow develops with downstream distance the boundary layer will eventually undergo transition to a turbulent state. Thus, at the leading edge and some distance downstream of it the flow is laminar before it rather abruptly transitions into a turbulent state.

A common practice when computing this type of flows is to use turbulence models tuned to perform well in *turbulent* flows in the entire domain considered. The reason is of course that it is not known in advance what regions of the flow will be laminar or turbulent. Unfortunately, the nature of a flow that somehow has undergone transition to a turbulent state is fundamentally different from that of a laminar flow. Therefore, it would be highly desirable to establish a computational procedure where laminar and turbulent regions can somehow be treated separately. One way do achieve this is to provide solutions to two sets of governing equations, one turbulent and the other laminar, in the full computational domain (note that they would have to be coupled). If we then can establish a measure of how turbulent the flow is (the intermittency factor) we may sift out the turbulent solution in turbulent regions and use the laminar solution in laminar regions. Further, in transitional regions ($0 < \gamma < 1$) we may obtain a fairly accurate prediction a more or less sophisticated blend of laminar and turbulent solutions.

3.2.1 The indicator function

At about the time Chen & Thyson (1971) published their results another branch of transition research was initiated by Libby (1974). He developed a theoretical model that allowed laminar and turbulent portions of fluid flows to be treated separately; the governing equations was *conditioned* into one laminar and one turbulent set of equations that are coupled using the intermittency concept described above. His objective was simply to derive an analogue to what experimentalists usually refer to as ‘conditioned sampling’. This experimental conditioning is performed by splitting a measured signal of arbitrary property, say ϕ, into periods that are either laminar or turbulent so that the experimentalist can determine, and distinguish between, properties the flow has when it is laminar or turbulent.

Libby imagined a sensor function, $I(x_i, t)$, that is unity if the flow at (x_i, t) is turbulent, otherwise zero. He then established time averages of the product of I with any other flow variable ϕ

$$\overline{\phi(x_i, t)I(x_i, t)} = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} \phi I dt \equiv \overline{\phi}_T(x_i) \tag{15}$$

where $\overline{\phi}_T(x_i)$ is the time average of ϕ taken during the time the flow is turbulent.

Similarly,

$$\overline{\phi(x_i,t)(1-I(x_i,t))} = \lim_{\tau\to\infty}\frac{1}{\tau}\int_0^{\tau}\phi(1-I)dt \equiv \overline{\phi}_L(x_i) \tag{16}$$

sifts out information on ϕ only when the flow is laminar. The two averages are related to the mean flow by

$$\overline{\phi} = \overline{\phi_L} + \overline{\phi_T} \tag{17}$$

Libby was among the first to suggest that I could be obtained from a transport equation on the form (the dynamics of the indicator function was established during the development of generalized functions)

$$\frac{\partial I}{\partial t} + \frac{\partial}{\partial x_j}(u_j I) = \dot{w} \tag{18}$$

with a source term $\dot{w}$, being the volumetric creation rate of I. By time averaging this equation Libby showed that the conditioned velocity components are not divergence-free

$$\frac{\partial \overline{u_{Tj}}}{\partial x_j} = \overline{\dot{w}} = -\frac{\partial \overline{u_{Lj}}}{\partial x_j} \tag{19}$$

where the last equality follows from Eqn 17 and use of the continuity equation. He was also the first to develop conditioned versions of other equations governing fluid flows, e.g. equations for Reynolds stresses and turbulence kinetic energy.

A few years later Dopazo (1977) and Byggstoyl & Kollmann (1981) suggested a slightly different formalism that appears, to the author, to be more consistent and will be used throughout the following description of the development of a conditionalized set of Navier-Stokes equations. Here the intermittency is defined as the probability of the point (x_i, t) being turbulent, i.e,

$$\gamma(x_i,t) = \langle I(x_i,t)\rangle \tag{20}$$

where $\langle\ \rangle$ denotes an *ensemble* average (which equals the time average in a statistically stationary flow).

In the derivation of the averaged conditioned (laminar and turbulent) equations the same function $I(x_i,t)$ as Libby used is needed. I is defined as

$$I(x_i,t) = \begin{cases} 0 & \text{if flow at } (x_i,t) \text{ is laminar} \\ 1 & \text{if flow at } (x_i,t) \text{ is turbulent} \end{cases} \tag{21}$$

The strategy for deriving the turbulent zone equations is to multiply the standard governing equations with this function, I, and thereafter taking an appropriate average of the resulting equation. For the laminar zone equations the procedure is the same with the exception that the governing equations are multiplied by $1-I$.

This procedure results in two sets of equations that allows to describe the flow at any point (x_i, t) in one way if the flow at (x_i, t) happens to be laminar and in another way if it is turbulent. If we can find a measure of how turbulent the flow is (the intermittency factor) we can combine solutions of these two sets of equations by use of this measure.

As we will see later a problem is the coupling between the laminar and turbulent zone equations. Consider for example the continuity equation. If a fluid particle undergoes transition during some period of time this will have the consequence that mass leaves the laminar zone and enters the turbulent one. In other words the continuity equation we are used to cannot be expected to hold in each zone separately (the laminar and turbulent zone velocities are not divergence free).

Another problem (related to the previous one) is that during the derivation correlations on the form $\langle I\partial\phi/\partial x_i\rangle$ and $\langle I\partial\phi/\partial t\rangle$ will appear. Evaluating these averages is necessary but will cause problems as the function I is discontinuous at laminar/turbulent interfaces and, as the interfaces moves, conserved quantities will transfer between the two zones.

Dopazo (1977) used the following identity

$$I\frac{\partial\phi}{\partial x_i} = \frac{\partial}{\partial x_i}(I\phi) - \phi\frac{\partial I}{\partial x_i} \tag{22}$$

Its average is

$$\left\langle I\frac{\partial\phi}{\partial x_i}\right\rangle = \frac{\partial}{\partial x_i}\langle I\phi\rangle - \left\langle \phi\frac{\partial I}{\partial x_i}\right\rangle = \frac{\partial}{\partial x_i}(\gamma\langle\phi_T\rangle) - \left\langle \phi\frac{\partial I}{\partial x_i}\right\rangle \tag{23}$$

where the average is taken over some small volume $\mathcal{V}$. The last term contains derivatives of the discontinuous function I and its average was shown by Dopazo to be given by

$$\left\langle \phi\frac{\partial I}{\partial x_i}\right\rangle = \lim_{\mathcal{V}\to 0}\left\langle \frac{1}{\mathcal{V}}\int_{S(x_i,t)}\phi n_i dS\right\rangle \tag{24}$$

where S is the laminar/turbulent interface surface and n_i a normal to this surface pointing towards the turbulent region. Thus we have

$$\left\langle I\frac{\partial\phi}{\partial x_i}\right\rangle = \frac{\partial}{\partial x_i}(\gamma\langle\phi_T\rangle) - \lim_{\mathcal{V}\to 0}\frac{1}{\mathcal{V}}\int_{S(x_i,t)}\langle\phi n_i\rangle\, dS \tag{25}$$

Unfortunately, the last term is either not well defined. Therefore, Steelant & Dick (1996) made some additional assumptions concerning the interface dynamics and derived the following relation

$$\left\langle I\frac{\partial\phi}{\partial x_i}\right\rangle = \overline{I\frac{\partial\phi}{\partial x_i}} = \gamma\frac{\partial\overline{\phi_T}}{\partial x_i} + \frac{1}{2}\left(\overline{\phi_T} - \overline{\phi_L}\right)\frac{\partial\gamma}{\partial x_i} \tag{26}$$

Note that Steelant & Dick (1996) used a time average (indicated by the overbars) where the time interval for integration was chosen to be large compared to turbulence time scales and small as compared with mean flow time scales. This expression for the (time) average of $I\partial\phi/\partial x_i$ is now expressed in known quantities and can easily be implemented in a numerical scheme. In order to compare with Dopazo's relation (in a statistically stationary flow) this equation can be written as

$$\begin{aligned}\overline{I\frac{\partial\phi}{\partial x_i}} &= \gamma\frac{\partial\overline{\phi_T}}{\partial x_i} + \frac{1}{2}\left(\overline{\phi_T} - \overline{\phi_L}\right)\frac{\partial\gamma}{\partial x_i}\\ &= \frac{\partial}{\partial x_i}\left(\gamma\overline{\phi_T}\right) - \overline{\phi_T}\frac{\partial\gamma}{\partial x_i} + \frac{1}{2}\left(\overline{\phi_T} - \overline{\phi_L}\right)\frac{\partial\gamma}{\partial x_i}\\ &= \frac{\partial}{\partial x_i}\left(\gamma\overline{\phi_T}\right) - \frac{1}{2}\left(\overline{\phi_T} + \overline{\phi_L}\right)\frac{\partial\gamma}{\partial x_i}\end{aligned} \tag{27}$$

Thus, for Eqn 25 to equal Eqn 27 (in a statistically stationary flow) we have

$$\lim_{\mathcal{V}\to 0}\frac{1}{\mathcal{V}}\int_{S(x_i,t)}\overline{\phi n_i}dS = \frac{1}{2}\left(\overline{\phi_T} + \overline{\phi_L}\right)\frac{\partial\gamma}{\partial x_i} \tag{28}$$

Similarly the $\overline{I\partial\phi/\partial t}$ term was found to be (Dopazo)

$$\left\langle I\frac{\partial\phi}{\partial t}\right\rangle = \frac{\partial}{\partial t}(\gamma\left\langle\phi_T\right\rangle) + \lim_{\mathcal{V}\to 0}\frac{1}{\mathcal{V}}\int_{S(x_i,t)}\left\langle\phi u_i^s n_i\right\rangle dS \tag{29}$$

(where u_i^s is the local velocity of the interface) or (Steelant and Dick)

$$\overline{I\frac{\partial\phi}{\partial t}} = \frac{\partial\left(\gamma\overline{\phi_T}\right)}{\partial t} - \frac{1}{2}\left(\overline{\phi_T} + \overline{\phi_L}\right)\frac{\partial\gamma}{\partial t} \tag{30}$$

With these tools for computing the different averages involving our special function I the conditioned equations can be derived. As an example the statistically stationary continuity equation is multiplied with I and averaged

$$\left\langle I\frac{\partial u_i}{\partial x_i}\right\rangle = 0 \tag{31}$$

With use of Eqn 25 we simply arrive at

$$\frac{\partial}{\partial x_i}(\gamma\left\langle u_{i,T}\right\rangle) = \lim_{\mathcal{V}\to 0}\frac{1}{\mathcal{V}}\int_{S(x_i,t)}\left\langle u_i n_i\right\rangle dS \tag{32}$$

which is the continuity equation for turbulent fluid. Similarly, with the approach of Steelant & Dick (1996) we obtain

$$\left\langle I\frac{\partial u_i}{\partial x_i}\right\rangle = \gamma\frac{\partial\overline{u_{i,T}}}{\partial x_i} + \frac{1}{2}\left(\overline{u_{i,T}} - \overline{u_{i,L}}\right)\frac{\partial\gamma}{\partial x_i} = 0 \tag{33}$$

and we clearly see, as mentioned above, that the zone velocities are not divergence free. Similar, but lengthier derivations of other conditioned equations, like momentum and Reynolds stress transport equations can be found in Steelant & Dick (1996), Byggstoyl & Kollmann (1981) or Dopazo (1977). For example, Byggstoyl & Kollmann (1981) derived equations for both laminar and turbulent zone velocities and for the turbulent zone k and ε equations suggested by Jones & Launder (1972) (note that the latter two equations first had to be conditioned). Several new unclosed terms appeared that had to be modelled before the conditioned model was applied to predict flows in plane jets and boundary layers. If it is assumed that the extra source terms that appear as a result of the conditional averaging can be closed the only problem remaining is that the intermittency function appearing in all the above equations is not known. One solution to this problem would be to use equations for the intermittency distribution similar to the one derived earlier in Section 3.1 (Eqn 14). That type of closures, however, require additional input, like spot propagation parameters, and is not suited for three-dimensional multi-purpose codes as they are formulated in terms of non-local quantities and are functions of the streamwise direction only. Obviously, a more elaborate closure of γ is needed.

3.2.2 A differential intermittency equation

Byggstoyl & Kollmann (1981) derived an expression for this function by considering a differential equation for the indicator function I

$$\frac{\partial I}{\partial t} + u_i \frac{\partial I}{\partial x_i} = V\delta(s) \tag{34}$$

where V is the velocity of the interface relative to the local fluid velocity, i.e.

$$Vn_i = u_i - u_i^s \tag{35}$$

An ensamble average of Eqn 34 yields

$$\left\langle \frac{\partial I}{\partial t} \right\rangle + \left\langle u_i \frac{\partial I}{\partial x_i} \right\rangle = \langle V\delta(S) \rangle \tag{36}$$

where the first term can by written as

$$\left\langle \frac{\partial I}{\partial t} \right\rangle = \frac{\partial \langle I \rangle}{\partial t} = \frac{\partial \gamma}{\partial t} \tag{37}$$

The third term is just averaged, i.e. equal to $\langle V\delta(S) \rangle$. The second term is more complicated but may be written as

$$
\begin{aligned}
\left\langle u_i \frac{\partial I}{\partial x_i} \right\rangle &= \left\langle \left(\langle u_i \rangle + u_i' \right) \frac{\partial I}{\partial x_i} \right\rangle \\
&= \left\langle \langle u_i \rangle \frac{\partial I}{\partial x_i} \right\rangle + \left\langle u_i' \frac{\partial I}{\partial x_i} \right\rangle \\
&= \langle u_i \rangle \left\langle \frac{\partial I}{\partial x_i} \right\rangle + \left\langle \frac{\partial u_i' I}{\partial x_i} \right\rangle - \left\langle I \underbrace{\frac{\partial u_i'}{\partial x_i}}_{=0} \right\rangle \\
&= \langle u_i \rangle \frac{\partial \gamma}{\partial x_i} + \frac{\partial \langle u_i' I \rangle}{\partial x_i} \\
&= \langle u_i \rangle \frac{\partial \gamma}{\partial x_i} + \frac{\partial \gamma \left\langle u_{T,i}' \right\rangle}{\partial x_i} \\
&= \langle u_i \rangle \frac{\partial \gamma}{\partial x_i} + \frac{\partial}{\partial x_i} \left(\gamma (1-\gamma) \left(\langle u_{T,i} \rangle - \langle u_{L,i} \rangle \right) \right) \qquad (38)
\end{aligned}
$$

In the last step above the following identity was used (Byggstoyl & Kollmann, 1981)

$$\left\langle u_{T,i}' \right\rangle = (1-\gamma)\left(\langle u_{T,i} \rangle - \langle u_{L,i} \rangle \right) \qquad (39)$$

Putting all terms together renders the intermittency equation of Byggstoyl & Kollmann (1981)

$$\frac{\partial \gamma}{\partial t} + \langle u_i \rangle \frac{\partial \gamma}{\partial x_i} = \frac{\partial}{\partial x_j} \left(\gamma (1-\gamma) \left(\langle u_{L,i} \rangle - \langle u_{T,i} \rangle \right) \right) + S_\gamma \qquad (40)$$

where

$$S_\gamma \equiv \langle V \delta(s) \rangle \qquad (41)$$

represents the mean entrainment of non-turbulent fluid into the turbulent zone and is the only term that requires additional modelling (even though Byggstoyl & Kollmann (1981) modelled the divergence term too). Dopazo (1977) arrived at the same equation even though he approached the problem somewhat differently.

In models involving a transport equation for the intermittency that are designed to predict also the onset of transition (e.g. the above model of Byggstoyl & Kollmann (1981) or the refined SLY intermittency model in Savill (2002*b*)) a weakness seems to be that γ source terms are modelled as being proportional to γ itself. Thus, any production of γ is dependent on the specification of a non-zero inlet value of γ, which is problematic as $\gamma = 0$ in laminar flows.

Steelant & Dick (1996) used a completely different approach when they derived an equation for the intermittency factor. They started off with the algebraic intermittency distribution suggested by Dhawan & Narasimha (1958), Eqn 14, and assumed that the equation holds along streamlines (it is not a function of the streamwise coordinate only). They arrived at

$$\frac{\partial \gamma}{\partial t} + \overline{u}_j \frac{\partial \gamma}{\partial x_j} = (1-\gamma)(\overline{u}_k \overline{u}_k)^{1/2} \beta(s) \tag{42}$$

where

$$\begin{aligned} \beta(s) &= 2f(s)f'(s) \\ f(s) &= \frac{as'^4 + bs'^3 + cs'^2 + ds' + e}{gs'^3 + h} \end{aligned} \tag{43}$$

with coefficients

$$\begin{gathered} a = \left(\frac{n\sigma}{U}\right)^{1/2} \quad b = -0.4906 \quad c = 0.204 \left(\frac{n\sigma}{U}\right)^{-1/2} \\ d = 0.0 \quad e = 0.04444 \left(\frac{n\sigma}{U}\right)^{-3/2} \quad g = 1.0 \quad h = 10e \end{gathered} \tag{44}$$

Here U is presumably the freestream velocity at the transition onset location and the streamline coordinate, s, is defined as

$$s = \int \frac{\overline{u}_i dx_i}{(\overline{u}_k \overline{u}_k)^{1/2}} \tag{45}$$

whereas $s' = s - s_t$, s_t being the transition location. Drawbacks of the model are that it relies on the assumptions behind the Dhawan & Narasimha (1958) distribution, that it is partly formulated in terms of a (non-local) streamwise coordinate and that it requires specification of both transition onset location and spot production rate. Therefore the purpose of introducing the γ equation is here not to improve the prediction of the location of transition but rather to improve the modelling of the effect of turbulent events once transition has begun. In Section 4.4 additional examples and references to intermittency related research will be given.

The spot production rate $\hat{n}\sigma$ appearing in the expressions for the model constants was closed using 'standard' algebraic correlations of which a few will be discussed in Section 4.2. For details the reader is referred to the original paper of Steelant & Dick (1996).

3.3 Low Reynolds number turbulence modelling

During the early 1970s the turbulence modelling community experienced a breakthrough when the classical paper of Jones & Launder (1972) on the first two-equation turbulence model was published. The very first two-equation models were so called high Reynolds number models that could not be appropriately integrated all the way down to solid walls as they resulted in erroneous wall normal behavior. Therefore it wasn't long before these models were complemented with specific functions, damping functions, designed to improve the near wall modelling in order to have profiles of important turbulence quantities (e.g. $\langle uv \rangle$ and k) that better resembled experimental data. Initially, this near wall modelling was aimed at improving predictions of fully turbulent boundary layers only.

However, the researchers of the time soon realized that these low Reynolds number extended models might also have a potential of predicting transitional flows. Imagine for example a boundary layer developing over a flat plate with a sharp leading edge. Further assume that there somehow exist a disturbance in the flow that is large enough to trigger the production of turbulence kinetic energy.In the turbulence models we look at the production rate is given by

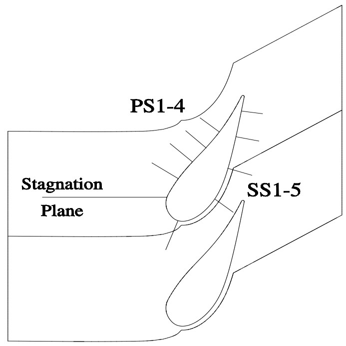
$$P_k = \underbrace{C_\mu \frac{k^2}{\varepsilon}}_{\nu_t} S_{ij} S_{ij} \tag{46}$$

From this equation it is clear that for production to occur there must exist an eddy-viscosity (the above-mentioned disturbance) and a varying mean velocity field so that the magnitude of the strain rate tensor is non-zero.

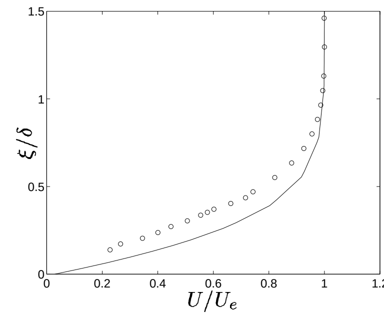
Just downstream of the leading edge in our flat plate test case the boundary layer is very thin. Thus, mean velocity gradients occur only in a narrow region in the close vicinity of the plate wall. Here, very close to the wall, low Reynolds number models give large values of dissipation of turbulent energy, ε, so that the viscous sublayer in a fully turbulent boundary layer can be accurately predicted. For the same reason a large portion of the turbulent kinetic energy produced in the (thin) pretransitional boundary layer immediately transforms into heat due to modelled viscous effects. However, the laminar diffusion will always transfer momentum from the mean flow towards the wall and thus, the boundary layer will grow and eventually reach a location where the modelled dissipation no longer balances the production of turbulent kinetic energy. Downstream of this location the growth of k will accelerate as, from Eqn 46, an increase in k will further increase its production, which eventually leads to a laminar to turbulent breakdown. Once k has started to grow the predicted transition process is usually rapid but is of course dependent on the exact formulation of the low Reynolds number model.

Typically, low Reynolds number models predict transition to occur to early, especially in stagnation point flows where the turbulence kinetic energy production is often overpredicted. An example of this failure of eddy-viscosity based closures is shown in Figure 7. Here boundary layer profiles on the suction side of one of the stator vane flows presented in Radomsky (2000) are compared with the results of a $v^2 - f$ model computation. Clearly, the predicted boundary layer transitions too soon.

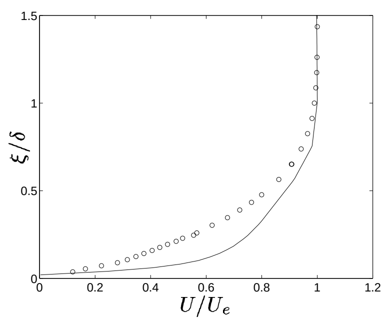
This approach to model transition is probably the most convenient one to use. The method is simply to apply the (low-Reynolds number) turbulence model be-



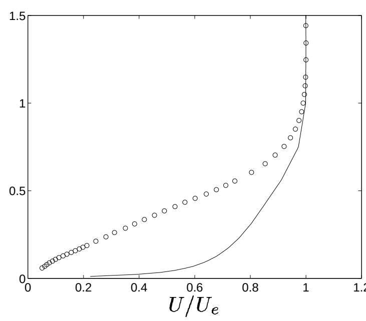
(a) Plotting locations



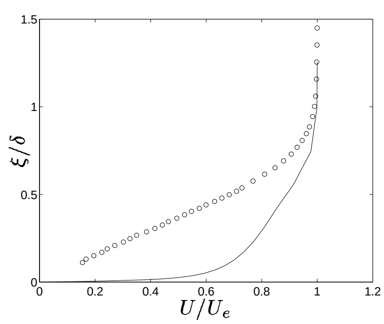
(b) SS1, $s/C = 0.21$



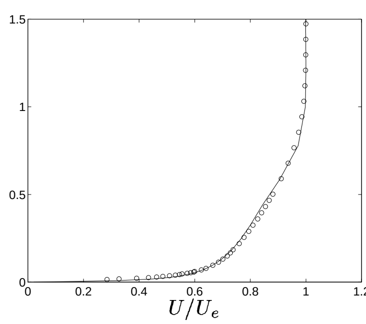
(c) SS2, $s/C = 0.50$



(d) SS3, $s/C = 0.75$



(e) SS4, $s/C = 1.0$



(f) SS5, $s/C = 1.2$

Figure 7: Development of the stator vane suction side boundary layer. ○: Experiment; —: FLUENT $\overline{v^2} - f$. U_e denotes the local freestream velocity and ξ is a wall normal coordinate.

lieved to be the most suitable for the particular flow and hope that it will predict the properties of the transitional part of the boundary (shear) layer just as well as the fully turbulent part. Indeed, this method would be the by far most tractable way of computing transitional flows as it would not matter whether the boundary layer is laminar (with freestream turbulence), transitional or fully turbulent; the same modelling would be applicable to all three kinds of flows. This is most likely the reason why there has been a considerable amount of work aiming at finding a turbulence model that has the above-mentioned properties.

One drawback of the method is that for transition to occur some initial source of turbulence is required. Hence, if ν_t is initially zero there cannot be any production of turbulence, nor can any transition ever take place. Therefore, assuming that the level of turbulence activity in the laminar part of the boundary layer is small ($\nu_t \approx 0$), turbulence energy must diffuse, or alternatively be convected, into the boundary layer from a freestream region for transition to possibly take place. This is, the mechanism is not very different from that of 'by-pass' transition, which suggests that by-pass transition is relatively well suited for this modelling approach. Another consequence is that natural transition cannot be predicted by use of low-Reynolds turbulence models.

In the quest of finding a turbulence model suitable for predicting by-pass transition a multitude of existing models have been explored and new models with improved transition-specific properties treatments have been developed. In an effort to systematically investigate the state-of-the-art of transition modelling a Transition Special Interest Group was initiated in the 1990's. This is a European project coordinated by Prof. Savill presently at the Cranfield Univeristy. A subproject within this consortium has been to evaluate most existing low Reynolds turbulence model formulations in order to determine the performance that can be expected from these models in transitional flows and to provide guidelines on how to obtain the best results in flows where transition is important. Note that different levels of turbulence closures, ranging from one-equation models to more advanced non-linear eddy-viscosity and Reynolds stress models have been examined. In Savill (2002*a*) results of no less than 22 models are evaluated.

Among their findings is that models employing damping factors of Launder-Sharma type (i.e. the D term in $\varepsilon = \tilde{\varepsilon} + D$ and $f_\mu = f_\mu(R_t)$) in general give the best predictions of mean flow quantities. Unfortunately these models require high grid resolution and are also sensitive to both inlet and freestream boundary conditions that many times are not very well defined. Therefore they suggested that future research should focus on developing methods that offer greater predictive abilities and are less demanding in terms of CPU power. As a possible candidate intermittency transport modelling coupled with a suitable turbulence model is mentioned.

Some authors have performed additional comparisons of low Reynolds turbulence/transition models that might be of interest. One is that of Biswas & Fukuyama (1994). They compared five low-Reynolds number $k - \varepsilon$ models (including one of their own) in flat plate and turbine blade computations performed with a boundary layer code. They concluded that the model versions with damping function expressed in terms of the turbulence Reynolds number are most likely to perform well. Westin & Henkes (1997) compared the performance of two differential Reynolds stress models, the Savill-Launder-Younis (SLY) model and the model of Hanjalić *et al.* (1995), with the $k - \varepsilon$ model of Launder & Sharma (1974). They found that the Reynolds stress closures had difficulties matching the performance of the less sophisticated $k - \varepsilon$ model. Worth noting is that Westin & Henkes (1997) discusses the influence of domain extensions on the computed results in flat plate computations and the approach taken seems reasonable. For example, the inlet boundary condition is specified upstream of the leading edge, and the wall opposite to the flat plate is placed sufficiently far away from the plate so that the results become independent of its actual location. The same approach was used by Craft *et al.* (1997), who compared a nonlinear eddy-viscosity model with the Launder-Sharma model, and Sveningsson (2005). Schmidt & Patankar (1991) on the other hand, specified profiles of turbulence quantities at the leading edge of the plate, or just downstream of it, and found a strong sensitivity to the profiles specified. Some additional 'low Reynolds number' transition modelling related work is presented in Section 4.1

A problem when computing transition with typical RANS turbulence models is the sensitivity of the predicted transitional behavior to the numerics involved. Even worse is that results predicted by certain RANS models in transitional flows have by Rumsey *et al.* (2005) been shown to depend arbitrarily on initial conditions. They analyzed the turbulence equations in a homogeneous shear flow from a dynamical systems point of view and were able to establish criteria for turbulence models to be able to reliably (non-arbitrarily) predict transitional phenomena.

3.4 A correlation based transport equation

Algebraic models suffer from the weakness of being cumbersome to implement in modern multipurpose CFD-schemes. The problem is that they often relate transition parameters, e.g. γ or the location of transition onset, to flow parameters, such as the momentum thickness, that are awkward to compute in a general manner. Therefore, they are used mostly in in-house codes where the user knows in advance, for example, how the grid was constructed and can adjust the modelling to suit a certain flow. Such flexible use becomes more or less impossible as the complexity of the problem considered increases.

On the other hand, the algebraic correlations contain valuable information on the behavior of transitional flows that is based on numerous experiments in a range of flows. Therefore, it would be desirable to incorporate this experimentally obtained information in the transition model. This is especially true since predictions of by-pass transition using standard low-Re turbulence models have proven to be very sensitive to numerical aspects like grid resolution and discretisation and would most likely profit from some guidance on where transition shall take place.

For transition models that solve a differential equation for the intermittency factor the situation is the opposite. Here the implementation, once the unknown source terms has been modelled, is usually straightforward as solving transport equations already is what CFD is all about. However, the intermittency model needs to be triggered by some source that is not too sensitive the above-mentioned numerical aspects.

In an effort to combine the best parts of two worlds Menter *et al.* (2004) developed a transition model based on transport equations for both the intermittency factor and a parameter that carries information on the location of transition onset – a transition onset Reynolds number. The latter is based on an, in principle arbitrary, algebraic correlation that relates the onset location to fluid properties that influence the transition process. The transition onset Reynolds number is then used to trigger production of intermittency.

The algebraic correlation used by Menter *et al.* (2004) involves only local flow quantities, with the exception of wall distance and a velocity defined at the edge of boundary layers. Therefore, the model is rather well suited for implementation in general purpose codes. It also includes corrections to improve predictions of transition in cases of separation induced transition. Unfortunately some rather important details of the model are considered proprietary and are at present not given. All other features of the model are given in Appendix A.

3.5 A transport equation for laminar kinetic energy

Yet another attempt to model transition stem from work of Mayle & Schultz (1997). They based their approach on the fact that the mechanisms behind the unsteadiness in a laminar boundary layer (the streaks in u velocity discussed in section 2.2) are different from that of 'normal' fluctuations in a fully turbulent boundary layer. In the latter additional (turbulent) kinetic energy is transferred from the mean flow by turbulent stresses acting on a mean velocity gradient. This production is then balanced by a transport of energy from so called energy containing structures to increasingly smaller scales and is eventually transformed into heat by viscous stresses at the very smallest scales of the flow. Contrary to this scenario, unsteadiness in laminar boundary layers is induced by freestream fluctuations.

Mayle & Schultz (1997) suggested that the production of this unsteadiness could be attributed a pressure-strain interaction that they in turn related to the splat mechanism[2] suggested by Bradshaw (1996). As mentioned in Section 2.2 Lardeau *et al.* (2005) proved this scenario erroneous as the pretransitional fluctuations were produced by a stress/mean strain mechanism and not pressure-diffusion, which actually proved to act as a sink. Consequently, as the only mean strain present is $\partial U/\partial y$ there must exists a $\overline{uv}$ correlation for the laminar kinetic energy to grow. This results is supported by the experimental study of highly disturbed freestream effects on pretransitional boundary layers by Volino & Simon (1994) and Volino & Simon (1997*b*). They measured significant levels of $\overline{uv}$ in the boundary layer prior to transition and it occurred at frequencies of the peak v' motion of the freestream unsteadiness. The levels of the correlated motions were not comparable to those of a turbulent boundary layer but large enough to influence the mean velocity and temperature profiles and thus able to cause elevations in skin friction and heat transfer.

Such a correlation can only exist in a Blasius boundary layer if fluid particles from the freestream penetrate the boundary layer and alter the streamwise momentum of the boundary layer. Consider again a chunk of fluid approaching the boundary layer. Most likely, it travels with a streamwise velocity (the freestream velocity on average) that is different from the streamwise velocity within the pretransitional boundary layer. Therefore, when the chunk of fluid enters the boundary layer it can be either decelerated as momentum is lost to the surrounding boundary layer fluid, which in turn causes $u\,'$ of the surrounding fluid to locally increase, or accelerated by gaining momentum from the boundary layer fluid. As the velocity on average is larger in the freestream the likelyhood that the boundary layer fluid is accelerated is larger than vice versa. This might explain why (Jacobs & Durbin, 2001) found that streaks of negative u' are more rare that those of positive u'. Once the u' fluctuation appears in the boundary layer the fluctuation grows and elongates in the streamwise direction under influence of the mean shear. Note that, as the flow in the boundary layer is not yet turbulent, the kinetic energy gained in the fluctuating u component is not passed on to any inertial subrange cascade, nor is it redistributed to any other (turbulent) fluctuating velocity component.

Returning to the work of Mayle & Schultz (1997) they derived a transport equation for laminar kinetic energy (LKE) in their effort to describe the effects of freestream turbulence on pretransitional boundary layers. From an order of

[2] The splat mechanism can be explained by considering a chunk of turbulent fluid that happens to move towards a solid wall (as it moves towards the wall it will contribute to an increase in the wall-normal Reynolds stress, $\overline{v^2}$). Due to the wall-normal momentum the chunk of fluid carries it will penetrate the boundary layer some distance before the presence of the wall will cause it to bounce back. The reflection of the fluid particle can only be accomplished by a local rise in pressure that also will act to push adjacent boundary layer fluid away from the point of interception. After that the particle has bounced away the boundary layer returns to essentially the same state as before the splat occured.

magnitude analysis they arrived at the following time-averaged equation for the LKE

$$\overline{u}\frac{\partial k_L}{\partial x} + \overline{v}\frac{\partial k_L}{\partial y} = \overline{u'\frac{\partial U'}{\partial t}} + \nu\frac{\partial^2 k_L}{\partial x^2} - \varepsilon_L \tag{47}$$

The second term on the right hand side is a standard diffusion term but the other two terms, i.e. the production and dissipation of LKE, respectively, require some further explanation. The first right hand side term arises from taking the average of $-u'\partial p'/\partial x$ (Mayle & Schultz, 1997). It represents production of laminar kinetic energy by work of pressure fluctuations [3], and thus, provides a link between the laminar fluctuations in the boundary layer and the turbulent freestream. This term has to be modelled and the following was suggested

$$\overline{-u'\frac{\partial p}{\partial x}} = \overline{u'\frac{\partial U'}{\partial t}} = C_\omega \frac{U_\infty^2}{\nu}\sqrt{k_L \cdot k_\infty}e^{-y^+/c^+} \tag{48}$$

C_ω, which is not constant, is believed to depend on turbulence spectra characteristics and on how well the freestream turbulence couples with the boundary layer. A concept that the boundary layer is especially receptive to certain frequencies (presumably related to some fraction of the Kolmogorov length scale) is established and used when modelling C_ω, which finally was closed using

$$C_\omega = C\left(\frac{\nu}{U_\infty}\right)^{2/3} Re_\Lambda^{-1/3} \tag{49}$$

where Re_Λ requires knowledge of the turbulence spectra. C and c^+ are assumed to be constant and taken as 0.07 and 13, respectively.

The (near wall) dissipation rate of the laminar kinetic energy was modelled with the in turbulence models frequently used

$$\varepsilon_L = 2\nu\frac{k_L}{y^2} \tag{50}$$

However, work of Thacker *et al.* (1999) suggests that the dissipation rate of laminar disturbances might by larger than for turbulent dissipation.

Lardeau *et al.* (2004) adopted the above concept (despite the fact that they questioned the physical reasoning behind it) and coupled a nonlinear eddy-viscosity model with the laminar kinetic energy equation of Mayle & Schultz (1997) via a slightly modified algebraic intermittency distribution originally suggested by Dhawan & Narasimha (1958), Eqn 14. The total kinetic energy was computed as

$$k = (1-\gamma)k_L + \gamma k_T \tag{51}$$

[3]Note that this term usually is rewritten as the sum of a pressure strain-rate term, $\overline{p'\partial u'_j/\partial x_j}$ and the divergence term, $\partial\overline{p'u'_j}/\partial x_j$ called the pressure diffusion term. The first is equal to zero in incompressible flows and the second term can only redistribute energy, and thus, not contribute to the total kinetic energy production (a consequence of Gauss' theorem).

where k_L is governed by Eqn 47 and the eddy-viscosity formula was modified according to

$$\nu_t = f_\mu c_\mu \frac{k\,(\gamma k_T)}{\varepsilon} \tag{52}$$

Finally, a damping function, f_ω, already used in the turbulence model, was introduced in the intermittency function

$$\gamma = 1 - f_\omega \exp\left[-\left(x - x_t\right)^2 \hat{n}\sigma \frac{U_{in}^2}{\nu^2}\right] \tag{53}$$

It is given by

$$f_\omega = \exp\left[-\left(\frac{y^*}{A}\right)^2\right] \tag{54}$$

where $y^* = u_\varepsilon y/\nu$ is a wall distance expressed in Kolmogorov variables.

What might be seen as a a drawback of the implementation in Lardeau *et al.* (2004) is that it involves an algebraic correlation to determine the onset of transition. Upstream of this location the linear term, which usually is the most important one, in the Reynolds stress closure has no effect on mean flow quantities as, given that $\gamma = 0$, the eddy-viscosity here is zero (cf. Eqn 52). Therefore, the added complexity in turbulence modelling of the nonlinear eddy-viscosity model can be questioned. Furthermore, the nonlinear model (Abe *et al.*, 2002), to which the transitional treatment was added, does not handle the stagnation point anomaly on its own. However, with the modification of the linear term this problem is solved (note that the nonlinear terms of the model do not contribute to the production of turbulence kinetic energy).

Walters & Leylek (2004) extended the concept of describing pretransitional boundary layers with a transport equation for laminar kinetic energy. They formulated a complete single-point RANS turbulence model that consists of not only the LKE (k_L) equation but do also include equations for turbulence kinetic energy (k_T) and a so called far-field dissipation rate ε.

The main concept of the model is to separate different types of fluctuating energies from each other. For example, as discussed above, the induced fluctuating energy in a laminar boundary layer should not be treated as 'normal' turbulence but still has to be taken into account as it causes both transition and some elevation of heat transfer in laminar boundary layer with a disturbed freestream. Further, in the development of the new model, the effect of small scale turbulence is separated from the effect of eddies of larger scales. The smaller scales, of energy $k_{T,s}$, contribute to production of additional turbulence as usual, whereas the larger scales, of energy $k_{T,l}$, are assumed to contribute to the production of laminar kinetic energy through the splat mechanism. The obvious question is at which eddy size the cutoff between the two types of deviations from the mean field should be placed. Walters & Leylek (2004) used an effective length scale, λ_{eff}, to divide the spectra

$$\lambda_{eff} = \min\left(C_\lambda d, \lambda_T\right) \tag{55}$$

where λ_T is the usual turbulence length scale $\lambda_T = k^{3/2}/\varepsilon$ and d is the wall distance. By assuming that the Kolmogorov inertial subrange applies over all wave numbers greater than $1/\lambda_T$, the small and large scale turbulent energies can then be obtained from

$$\begin{aligned} k_{T,s} &= k_T\left(\frac{\lambda_{eff}}{\lambda_T}\right)^{2/3} \\ k_{T,l} &= k_T\left(1-\left(\frac{\lambda_{eff}}{\lambda_T}\right)^{2/3}\right) \end{aligned} \tag{56}$$

Note however that, as shown by Sveningsson (2005), the use of these equations to separate small scale energy from large scale energy makes the model very sensitive to the freestream length scale. All other details of the model are given in Appendix B.

4 Additional transition models

Collected here are several models aimed at improving the predictions of transitional flows. Included are references to a few low Reynolds number models specifically designed for transition, some commonly used algebraic correlations and a few transport equations for the intermittency factor suggested by various authors. It is of course impossible to include all relevant earlier studies but it is the authors' hope that this section provides an overall picture of earlier ideas in the transition modelling area. Included are, for example, some ideas on how to sensitize (RANS) transition models to separation induced and natural transition.

4.1 Low Reynolds Number Formulations

One study that needs to be mentioned here is that of Wilcox (1994). It constitutes an effort to display the interaction of the model constants in the k equation and the length scale determining equation (here ω). A constraint on the model constants, corresponding to the $C_{\varepsilon 1}$ and $C_{\varepsilon 2}$ of the $k-\varepsilon$ model, for transition to possibly occur is derived. Based on the findings from an analytical study of the model equations a new set of damping functions ('viscous modifications') was suggested. The modified model was designed mainly to predict the flow in transitional and turbulent regions and not intended for predictions of the actual location of transition. The location of transition was in fact 'controlled' by altering the freestream turbulence level or, which is more interesting, altering the wall boundary condition of the ω equation to simulate a strip of surface roughness. Note that Wilcox (1975) and Wilcox (1981) also sensitized a turbulence closure (a $k-\omega^2$ model) to predict the actual location of transition onset and, in particular, its sensitivity to flow parameters such as pressure gradients. This was done by introducing a functional dependence of turbulence model constants on the turbulence Reynolds number and an additional coefficient obtained from linear stability computations.

Rademehr & Patankar (2001) developed a model aimed at improving predictions of transition in gas turbine applications. The model uses local quantities only and is similar to the Launder-Sharma model, with somewhat different damping functions. The model does a reasonably good job in predicting the ERCOFTAC T3A and T3B cases (flat plate with 3 and 6% FSTI, respectively). The model is shown to perform better than the Launder-Sharma model, especially in the turbulent region where the latter underpredicts C_f. Note that in all flat plate computations profiles of k and $\tilde{\varepsilon}$ where specified in the *laminar* part of the boundary layer. In contradiction to what has been found in other studies (e.g. Craft *et al.* (1997), Schmidt & Patankar (1991)) the results are claimed to be insensitive to these profiles. The influence of acceleration on the model's predictions of transition is also examined. For this purpose data of Blair (1982), with constant acceleration, are used. In general, the agreement for both location and length of transition is good. However, in the fully turbulent region Stanton numbers are underpredicted. Finally, the model was tested on a turbine blade case with 4% FSTI. Here too the agreement with measurements is good. Somewhat surprisingly, the model seems to be able to fairly accurately predict stagnation region heat transfer without any attention being paid to the stagnation point anomaly that usually has a great effect on this type of computations.

In a series of studies of Hassan and co-workers (e.g. Warren & Hassan (1997*a*), Warren & Hassan (1997*b*), Warren & Hassan (1998) and McDaniel & Hassan (2000), where the latter deals with by-pass transition) developed measures to treat nonturbulent fluctuations within the framework of traditional RANS modelling. The fluctuations can for example be Tollmien-Schlichting waves of the streaky pattern seen in by-pass transition (cf. Figures 1 and 2). The approach involves a $k-\zeta$ turbulence model[4], the computation of a correlation for a nonturbulent viscosity and the classical Dhawan & Narasimha (1958) correlation for the intermittency factor to blend the turbulent and nonturbulent viscosities. As the model involves correlations it cannot be regarded as strictly 'predictive' but McDaniel & Hassan (2000) show fairly accurate computations of seven (of a total of eight) ERCOFTAC T3 test cases (zero and nonzero pressure gradient flows) and needs to be mentioned. Also worth noting is that the method (without the by-pass part of the model) was tested by Warren & Hassan (1998) in both a boundary layer code

[4] ζ is the modelled enstrophy (the variance of vorticity)

and in a more general Navier-Stokes code.

4.2 Some algebraic correlations

One of the most extensively used algebraic correlations is the one suggested by Abu-Ghannam & Shaw (1980). They started with a correlation of Hall & Gibbings (1972),

$$Re_{\theta,tr} = 190 + \exp\left[6.88 - 1.03\,Tu\right] \tag{57}$$

which was adjusted to more recent data giving

$$Re_{\theta,tr} = 163 + \exp\left[6.91 - \,Tu\right] \tag{58}$$

that was assumed to be valid for zero-pressure gradient flows. Finally the range of applicability of the correlation was extended to pressure gradient flows by introducing a function $f(\lambda_\theta)$ according to

$$Re_{\theta,tr} = 163 + \exp\left[f(\lambda_\theta) - \frac{f(\lambda_\theta)}{6.91}\,Tu\right] \tag{59}$$

$$f(\lambda_\theta) = 6.91 + \begin{cases} 12.75\lambda_\theta + 63.64\lambda_\theta^2 & (\lambda_{\theta_{tr}} \leq 0) \\ 2.48\lambda_\theta - 12.27\lambda_\theta^2 & (\lambda_{\theta_{tr}} > 0) \end{cases} \tag{60}$$

This dependence of f on the pressure gradient parameter, $\lambda_\theta = (\theta^2/\nu)dU_\infty/dx$), gives that the effect of an adverse pressure gradient in promoting transition is greater than the retarding effect of a favorable gradient. Abu-Ghannam & Shaw (1980) also modified the intermittency distribution suggested by Dhawan & Narasimha (1958). They suggest

$$\gamma(x) = 1 - \exp\left[-5\eta^3\right] \tag{61}$$

where η is a non-dimensional distance from the start of transition

$$\eta = \frac{Re_x - Re_{x,t}}{Re_{x,end} - Re_{x,t}} \tag{62}$$

The following estimate for the end of transition was given (note that they use the end of transition instead of a spot production rate to characterize the transitional region)

$$Re_{x,end} = Re_{x,t} + 16.8\;Re_{x,t}^{0.8} \tag{63}$$

Mayle (1991) studied the role of transition in gas turbine engines from both a theoretical and experimental point of view and presented the status of transition modelling in the early 1990's.

According to Mayle the onset of transition in a gas turbine-like environment is largely determined by the amount of freestream disturbance (turbulence or wake induced), whereas the spot production rate seems to be controlled by the pressure gradient at transition onset. Other phenomena that are known to affect transition are usually less important in gas turbine applications. Further, it was suggested that transition due to separation is controlled by the momentum thickness at separation and also by the parameters that affect the length of the laminar shear layer in the separation bubble.

Mayle (1991) collected all suitable experimental data available at the time and tried to correlate the location of transition onset, here represented by the transition onset Reynolds number $\mathrm{Re}_{\theta t}$, with turbulence intensity. In the case of zero pressure gradient a fit to measured data gave

$$\mathrm{Re}_{\theta t} = 400\, Tu^{-5/8} \tag{64}$$

This relation matched the data rather well for a wide span of turbulence intensities and is a commonly used onset correlation. A similar correlation was obtained for the spot production rate

$$\hat{n}\sigma = 1.5 \times 10^{-11}\, Tu^{7/4} \tag{65}$$

Together with Eqn 14 the two above closure models may now be used to account for the effects of transition. Due to its simplicity this approach has been used extensively in gas turbine design, especially in two-dimensional boundary layer codes where the formulation in terms of non-local quantities (θ, Tu_∞) is not a problem. For example it was used by Steelant & Dick (1996) in the zero pressure gradient flows they computed even though they used slightly different model constants (420, -0.69 and 1.25×10^{-11}) and used a definition of transition onset that differs somewhat from the standard procedure. For pressure gradient flows Steelant & Dick (1996) suggested the following relation for the spot production rate:

$$\frac{\hat{n}\sigma}{(\hat{n}\sigma)_{\mathrm{ZPG}}} = \begin{cases} (474\, Tu^{-2.9})^{1-\exp[2\cdot 10^6 K_t]} & (K_t < 0) \\ 10^{-3227 K_t^{0.5985}} & (K_t > 0) \end{cases} \tag{66}$$

It should be noted that the correlations do not involve any measure of the scale of turbulence. Some evidence indicating that it is important was given. For example it was found that exchanging the grid used to produce turbulence resulted in a $\pm 25\%$ change in streamwise location of transition even though the turbulence level remained the same.

Mayle also discussed the effect of pressure gradients. In this case the amount of data was sparse but suggested that in accelerating flow the spot production decreases as the acceleration increases (it should approach zero as the relaminarization limit is approached, $K_t \approx 3 \times 10^{-6}$). However, the influence of the acceleration parameter, K_t, on spot production is weak. For decelerating flows on the other hand, the effect is rather large, especially if the turbulence intensity is low.

The effect of acceleration on transition onset was found to be important whenever the turbulence activity in the freestream is low. However, for levels typical for gas turbine engines the effects was considered negligible. That is, given that the turbulence intensity is sufficiently high, it is the only parameter that affects transition onset and Eqn 64 is assumed to be valid.

Finally, Mayle (1991) suggests some correlations for separation induced transition. They are expressed in terms of a difference in Reynolds number between the point of separation (s) and the onset of transition (t), i.e.

$$Re_{x,st} = Re_{x,t} - Re_{x,s} = \begin{cases} 300 Re_{\theta s}^{0.7} & \text{(short bubbles)} \\ 1000 Re_{\theta s}^{0.7} & \text{(long bubbles)} \end{cases} \tag{67}$$

Later Malkiel & Mayle (1996) performed measurements of transition in a separation bubble that corroborated this correlation for low freestream turbulence levels. For freestream turbulence levels typical of turbomachinery, however, additional data in needed to check its validity.

Thermann *et al.* (2001) demonstrated the potential of including an algebraic transition model in a standard CFD solver. They performed both two and three-dimensional computations of two compressor blades, one being subsonic and the other transonic. Great improvements in skin friction predictions as compared with 'fully turbulent' (using the Chien turbulence model) computations were found.

A few different criteria for transition onset and intermittency functions were investigated. Different transition models are used depending on whether transition is expected to occur in a separated or attached flow region. For separated-flow transition the short bubble correlation in Eqn 67 was used to define transition onset. Downstream of the onset location they used the same intermittency distribution as was used by Walker *et al.* (1988), namely the Chen & Thyson (1971) function, which reads,

$$\gamma(x) = 1 - \exp\left[-n|_{tr}(x - x_{tr}) \int_{x_{tr}}^{x} \frac{1}{u} dx'\right] \tag{68}$$

but with the modified spot production rate:

$$\hat{n}\sigma = \frac{1}{40} Re_{x,tr}^{-1.34}, \qquad \hat{n} = n\frac{\nu^2}{u_a^3} \tag{69}$$

For by-pass transition the correlation of Solomon *et al.* (1996), which is an extension of the Chen & Thyson (1971) function, was used to determine the intermittency distribution. It involves the spot spreading half-angle α and the spot propagation parameter σ as functions of the local pressure gradient λ_θ according to

$$\gamma(x) = 1 - \exp\left[-n|_{tr} \int_{x_{tr}}^{x} \frac{\sigma}{\underbrace{\tan \alpha u}_{?}} dx' \int_{x_{tr}}^{x} \tan(\alpha) dx'\right] \tag{70}$$

where

$$
\begin{aligned}
\alpha &= 4.0 + \frac{22.14}{0.79 + 2.71 \exp\left[47.63\lambda_\theta\right]} \\
\sigma &= 0.03 + \frac{0.37}{0.48 + 3.0 \exp\left[52.9\lambda_\theta\right]} \\
N &= 0.86 \cdot 10^{-3} \exp\left[-0.564 \ln(Tu)\right] f(Tu, \lambda_{\theta_{tr}}) \\
N &= n\frac{\sigma\theta^3}{\nu} \\
f(Tu, \lambda_{\theta_{tr}}) &= 0 + \begin{cases} \exp\left[2.134 \ln(Tu)\lambda_{\theta_{tr}} - 59.23\lambda_{\theta_{tr}}\right] & (\lambda_{\theta_{tr}} \leq 0) \\ \exp\left[-10\lambda_{\theta_{tr}}^{0.5}\right] & (\lambda_{\theta_{tr}} > 0) \end{cases} \\
\lambda_\theta &= \frac{\theta^2}{\nu}\frac{du}{dx}
\end{aligned}
\tag{71}
$$

This correlation was also used by Byvaltsev & Kawaike (2005) who compared its performance with the slightly modified version of the same correlation suggested by Byvaltsev & Nagashima (2001). Finally, the Sieger *et al.* (1995) criterion was used to determine the onset of by-pass transition;

$$
\begin{aligned}
Re_{\theta,tr} &= \exp\left[5.094 - 25\left(\frac{\overline{Tu}}{100}\right)^{1.6}\right] \\
&- 400\exp\left[-0.01\left(\overline{Tu} + 2.5\right)^4\right] \\
&+ \exp\left[f(\lambda_\theta) - \frac{f(\lambda_\theta)}{6.91}\overline{Tu}\right]
\end{aligned}
\tag{72}
$$

where $\overline{Tu}$ is the average of the turbulence levels calculated with the inflow velocity and the local velocity. Note that the third term on the right hand side is the Abu-Ghannam & Shaw (1980) correlation. γ was now used to modify the eddy-viscosity but only in the boundary layer. Outside a boundary layer interface γ was set to unity. For the highly tree-dimensional subsonic computations the two models above were combined to allow for both by-pass transition and transition induced by a laminar separation bubble. This improved predictions close to endwalls where strong secondary flows were present.

A similar study was carried out by Ameri & Arnone (1996) who also included an algebraic transition model in turbine rotor blade computations. They used the Baldwin-Lomax mixing length model for turbulence closure and tried to model the effect of (time-averaged) wake induced intermittency. They followed the transition model directives suggested by Mayle (1991). In particular they used Eqn 14 for the intermittency distribution, Eqn 64 for transition onset and Eqn 65 to model the spot production rate in zero pressure gradient regions. For non-zero pressure gradients, Mayle's graphical correlation was somehow employed. Note that the intermittency function was expressed in terms of a streamline coordinate (restricted to walls) that for the profile was assumed to follow the profile surface (the spanwise velocity was assumed to be zero).

The transition model was coupled to the turbulence model according to

$$\mu_t = f\mu_{t,ew} + (1-f)\,\mu_{t,bl}\tilde{\gamma}(s) \tag{73}$$

where f is a weighting function necessary due to the turbulence model formulation, which gives one *endwall* viscosity, $\mu_{t,ew}$ and one blade viscosity, $\mu_{t,bl}$. Finally, $\tilde{\gamma}(s)$ is obtained from

$$\tilde{\gamma}(s) = 1 - (1 - \gamma_n(s)) \tag{74}$$

Details of how μ_t is computed in the freestream and the boundary layer are not given.

The results in this paper are in general in good agreement with experimental data. Stanton number contours are close to measured values on both sides of the rotor and even better agreement is found on the rotor endwall. This is especially interesting as the modelling of turbulence was not that sophisticated. This suggests, as a simple mixing length model is not expected to do a very good job predicting the turbulence field, that this flow, or at least the heat transfer, is not very sensitive to how well the turbulence is captured.

Papanicolaou & Rodi (1999) also used a correlation-based approach to guide the two-layer $k-\varepsilon$ turbulence model (cf. Rodi, 1991) they used in two transitional flows – a long body with circular leading edge and a backward facing step, i.e. two flows with expected separation induced transition (the two-layer model was also used by Cho *et al.* (1993) to compute (wake-induced) transitional attached turbine cascade flows). The freestream Tu varied between 0.17 and 5.56 in the first case and was on the order of 10^{-2} in the second case. The intermittency function used, based on the work of Chen & Thyson (1971), is

$$\gamma(x) = 1 - \exp\left[-G(x - x_{tr})\int_{x_{tr}}^{x} \frac{dx}{U_\infty}\right] \tag{75}$$

where

$$G = \frac{\exp\left[0.99\,Tu\right]}{100}\frac{U_\infty^3}{\nu^2} Re_{\theta,tr}^{-8/3} \tag{76}$$

and

$$Re_{\theta,tr}^2 = \left(1 + \frac{0.05}{\exp\left[0.365\,Tu\right]}\right) Re_{\theta,sep}^2 + \frac{17,000}{\exp\left[0.509\,Tu\right]} \tag{77}$$

Here Re_θ is the local Reynolds number based on the momentum thickness. Subscripts *tr* and *sep* refer to its value at the point of transition onset and the point of separation, respectively. The location of transition onset was taken as the point where Re_2 exceeds the value of $Re_{\theta,tr}$. The computed intermittency factor was used to modify the eddy-viscosity ($\mu_t^{\mathrm{mod}} = \gamma\mu_t$). Note that this caused stability problems in the low Tu case laminar regions. The problem was solved by using $\gamma = 1$ in the turbulent diffusion model in laminar freestream regions and $\gamma \approx 0.1$ in laminar boundary layers. In general this turbulence/transition model was fairly accutare in predicting the range of test cases considered. It was found, however, that the model performed better at moderate levels of freestream turbulence ($0.6 < Tu < 5.6$) than at lower levels. A limitation of the model is of course its one-dimensional form of the intermittency correlation.

Suzen & Huang (2000) used a transition onset model to trigger the production in intermittency transport model given in the following section. The spot production rate $\hat{n}\sigma$ was for zero pressure gradient fbws determined from the Mayle (1991) correlation (with a slightly modified constant):

$$\hat{n}\sigma = 1.8 \cdot 10^{-11}\, Tu^{7/4} \tag{78}$$

For pressure gradient fbws the correlation suggested by Steelant & Dick (1996), Eqn 66, was used. The onset of transition was specified using the following correlation:

$$Re_{\theta_t} = (120 + 150\, Tu^{-2/3}) \coth(4.0(0.3 - K_t \cdot 10^5)) \tag{79}$$

“where K_t is the minimum value of the acceleration parameter in the downstream deceleration region” and Tu is the freestream turbulence intensity at onset of transition.

Later, Suzen *et al.* (2002) modified these correlations. For fbws subjected to pressure gradients the following correlation was used

$$\frac{\hat{n}\sigma}{(\hat{n}\sigma)_{\mathrm{ZPG}}} = \begin{cases} M^{\left[1-\exp[0.75\cdot 10^6 K_t\, Tu^{-0.7}]\right]} & (K_t < 0) \\ 10^{-3227 K_t^{0.5985}} & (K_t > 0) \end{cases} \tag{80}$$

where

$$M = 850\, Tu^{-3} - 850\, Tu^{-1/2} + 120 \tag{81}$$

and $(\hat{n}\sigma)_{\mathrm{ZPG}}$ is obtained from Eqn 78. $K_t = (\nu/U^2)(\mathrm{d}U/\mathrm{d}x)$ is evaluated at transition onset. Suzen *et al.* (2002) also encountered transition induced by fbw separation and used the following relation to determine transition onset:

$$Re_{s-t} = 2.5 \cdot 10^4 \log_{10} \coth(0.1732\, Tu), \tag{82}$$

As it is not known *a priori* whether the fbw will separate (before transition) or not, i.e. if Eqn 79 or 82 should be used, a prediction-correction scheme was used to compute transition onset. The approach was to first compute a laminar solution, find the laminar separation point and use Eqn 82 to determine transition onset, which was updated every 100 iteration. During this procedure the transition point moved upstream and the separation point downstream. If the process converged to a solution that satisfied Eqn 82 a final solution with separated-flow transition was assumed. The attached-flow correlation, Eqn 79, was used if the predicted separation point moved downstream of that of transition or if the (laminar) separation bubble disappeared.

Another algebraic correlation was suggested by Menter *et al.* (2004). It was used as part of a source term in a transport equation for the local transition onset momentum thickness Reynolds number in their transition model. It reads

$$Re_{\theta t} = \frac{\rho \theta U_0}{\mu} = 803.73 \left(Tu + 0.6067 \right)^{-1.027} F(\lambda_\theta, K) \tag{83}$$

where the function $F(\lambda_\theta, K)$ is given by

$$\begin{aligned} F(\lambda_\theta, K) &= 1 + \left(10.32\lambda_\theta + 89.47\lambda_\theta^2 + 265.51\lambda_\theta^3 \right) e^{Tu/3.0} \quad (\lambda_\theta \leq 0) && (84) \\ F(\lambda_\theta, K) &= 1 + F_2(K) \left(1 - e^{Tu/1.5} \right) + \\ &+ 0.556 \left(1 - e^{-23.9\lambda_\theta} \right) e^{Tu/1.5} \quad (\lambda_\theta > 0) && (85) \\ F_2(K) &= 0.0962K \cdot 10^6 + 0.148 \left(K \cdot 10^6 \right)^2 + 0.0141 \left(K \cdot 10^6 \right)^3 && (86) \end{aligned}$$

This correlation is clamied to retain the best properties of the Abu-Ghannam and Shaw correlation and the Suzen *et al.* (2002) correlation. Note that the set of equations are implicit in θ. To enhance numerical stability the correlation was limited as follows:

$$\begin{aligned} -0.1 \leq \; & \lambda_\theta \leq 0.1 \\ -3 \cdot 10^6 \leq \; & K \leq 3 \cdot 10^6 \\ & Re_{\theta t} \geq 20 \end{aligned} \tag{87}$$

Andersson *et al.* (1999) performed boundary layer computations to find the optimal type of disturbances that produce the largest growth of pretransitional laminar streaks. They found that almost any type of disturbance will develop into a streamwise streak and do also illustrate the so called lift-up effect. Based on these computations they proposed a correlation that proved to be quite successful in the freestream turbulence intensity range $1\% < Tu < 6\%$, which is the range where the streak breakdown process is expected to dominate. Their correlation is

$$\sqrt{Re_T}\, Tu = \text{const.} \approx 1.2 \times 10^3 \tag{88}$$

where the value of the constant was obtained from several different experimental sets of data. The constant for each set of data is given in Andersson *et al.* (1999), the value presented here is only an approximation by the present authors. The most appealing aspect of this correlation is that its derivation is based on a theoretical description of the mechanism that actually causes the transition.

4.3 Notes on computing boundary layer quantities

The by far greatest drawback with algebraic transition models is that the equations describing, for example, the onset location or the intermittency function, contain (non-local) parameters that are difficult to determine. This is especially true if the equation is to be used in a general-purpose code for arbitrary geometries. Given below are some hints, found in different references, on how to evaluate some frequently appearing boundary layer quantities.

4.3.1 Momentum thickness

In Ameri & Arnone (1996) Twaite's approximation was used when calculating the momentum thickness, θ. It reads,

$$\theta^2(s) = \frac{0.45\nu}{U_e^6(s)} \int_0^s U_e^5 ds' \tag{89}$$

Using this approximation neatly eliminates the requirement of determining the location of the edge of the boundary layer. Here ν is the freestream kinematic viscosity, s is the local surface distance from the leading stagnation point and U_e is the "edge velocity" (not yet clear what this means). The latter was computed using the blade surface static to total local pressure ratio. The integration was carried out along the streamwise grid lines, which was claimed to be acceptable as long as the inviscid streamlines are reasonably aligned with the grid lines.

4.3.2 Boundary Layer Thickness

Thermann *et al.* (2001) computed the boundary layer thickness for each time step using the 99% criterion.

4.3.3 Freestream Turbulence Intensity

Ameri & Arnone (1996) computed a turbine rotor blade configuration for which a freestream turbulence intensity of 3.5% had been documented at the inlet in experiments. The problem is that the transition model used requires specification of freestream turbulence along the edge of the boundary layer, Tu_e. This quantity is of course extremely difficult to treat strictly according to its definition as, again, the location of the edge of the boundary layer is hard to locate. Therefore Ameri & Arnone (1996) used the following relationship:

$$Tu_e \times U_e = \text{const} \tag{90}$$

4.3.4 Turbulent viscosity

Thermann *et al.* (2001) used the intermittency to modify the turbulent viscosity according to:

$$\mu_t^\star = \gamma \mu_t \tag{91}$$

The flow in the freestream was assumed to be fully turbulent and therefore the modification was performed inside the boundary layer only. At the interface a switching function was applied and γ set to unity outside the boundary layer.

4.3.5 Streamwise acceleration

The acceleration in the streamwise direction, $\partial(U_k U_k)^{1/2}/\partial s$ (in some models denoted simply as dU/dx), is involved in the flow acceleration parameter, K, and the pressure gradient parameter, λ_θ and can be computed as follows (Menter *et al.*, 2004)

$$\frac{\partial(U_k U_k)^{1/2}}{\partial s} = \frac{U_i}{(U_k U_k)^{1/2}} \frac{\partial(U_l U_l)^{1/2}}{\partial x_i} \tag{92}$$

$$\frac{\partial(U_k U_k)^{1/2}}{\partial x_i} = \frac{U_j}{(U_k U_k)^{1/2}} \frac{\partial U_j}{\partial x_i} \tag{93}$$

4.4 Intermittency transport equations

Cho & Chung (1992) suggested a k-ε-γ equation turbulence model that was developed in an attempt to address the anomalies of the plane/round jet and the plane jet/plane wake (it was not intended for transitional boundary layers). Some truly encouraging results were obtained for these flows. The model adopts ideas from the work on conditioned equations (even though it is formulated in one (turbulent) set of equations). One is (Chevray & Tutu, 1978) that there is an additional velocity scale related to the mean velocity field. A model of Lumley (1978) was used to account for its effect when modelling the eddy-viscosity. Another is the intermittency equation modelled as of Byggstoyl & Kollmann (1986). Finally, some additional terms were added to the dissipation rate equation before the above-mentioned flows were rather successfully computed. Unfortunately, the proposed model was never given a completely general formulation as no model for the normal components of the Reynolds stress tensor was given. Therefore, only their intermittency model is repeated here. It reads

$$
\begin{aligned}
u_j \frac{\partial \gamma}{\partial x_j} &= \frac{\partial}{\partial x_j}\left((1-\gamma)\frac{\nu_t}{\sigma_g}\frac{\partial \gamma}{\partial x_j}\right) + S_\gamma \\
S_\gamma &= C_{g1}\gamma(1-\gamma)\frac{P_k}{k} + C_{g2}\frac{k^2}{\varepsilon}\frac{\partial \gamma}{\partial x_j}\frac{\partial \gamma}{\partial x_j} - C_{g3}\gamma(1-\gamma)\frac{\varepsilon}{k}\Gamma \qquad (94) \\
\Gamma &= \frac{k^{5/2}}{\epsilon^2}\frac{U_i}{(U_k U_k)^{1/2}}\frac{\partial u_i}{\partial x_j}\frac{\partial \gamma}{\partial x_j} \qquad (95)
\end{aligned}
$$

Suzen & Huang (2000) developed a new intermittency factor transport equation aimed at combining the best properties of two intermittency models. The models are the Steelant & Dick (1996) model that was designed to reproduce the Dhawan & Narasimha (1958) (algebraic) distribution along the transitional region and the Cho & Chung (1992) model which has proven to give accurate cross-stream variations of γ.

The new model involves a blend of the production terms of the two earlier models mentioned above. The production terms, $T_0 - T_3$, are defined as

$$
T_0 = C_0 \rho (u_k u_k)^{1/2} \beta(s) \qquad (96)
$$

with $\beta(s)$ from Eqn 43 with recalibrated constants

$$
\begin{aligned}
&a = 50\left(\frac{n\sigma}{U}\right)^{1/2} \quad b = -0.4906 \quad c = 0.204\left(\frac{n\sigma}{U}\right)^{-1/2} \\
&d = 0.0 \quad e = 0.04444\left(\frac{n\sigma}{U}\right)^{-3/2} \quad g = 50 \quad h = 10e \qquad (97)
\end{aligned}
$$

and

$$
\begin{aligned}
T_1 &= C_1 \gamma \frac{P_k}{k} \qquad (98) \\
T_2 &= C_2 \gamma \rho \frac{k^{3/2}}{\varepsilon}\frac{u_i}{(u_k u_k)^{1/2}}\frac{\partial u_i}{\partial x_j}\frac{\partial \gamma}{\partial x_j} \qquad (99) \\
T_3 &= C_3 \rho \frac{k^2}{\varepsilon}\frac{\partial \gamma}{\partial x_j}\frac{\partial \gamma}{\partial x_j} \qquad (100)
\end{aligned}
$$

T_0 origins from Steelant & Dick (1996) whereas $T_1 - T_3$ are from the Cho & Chung (1992) model. The terms $T_0 - T_2$ are blended using

$$
\begin{aligned}
P_\gamma &= (1-F)T_0 + F(T_1 - T_2) \qquad (101) \\
F &= \tanh^4\left[\frac{k}{S\nu}\frac{1}{200(1-\gamma^{0.1})^{0.3}}\right] \qquad (102)
\end{aligned}
$$

The following diffusion model of γ is suggested

$$D_\gamma = \frac{\partial}{\partial x_j}\left([(1-\gamma)\gamma\sigma_{\gamma_l}\mu + (1-\gamma)\sigma_{\gamma_t}\mu_t]\,\frac{\partial \gamma}{\partial x_j}\right) \tag{103}$$

and the final form of the model reads

$$\frac{\partial \rho\gamma}{\partial t} + \frac{\partial \rho u_j \gamma}{\partial x_j} = (1-\gamma)P_\gamma + T_3 + D_\gamma \tag{104}$$

with model constants

$$\sigma_{\gamma l} = \sigma_{\gamma t} = C_0 = 1.0 \quad C_1 = 1.6 \quad C_2 = 0.16 \quad C_3 = 0.15 \tag{105}$$

The intermittency factor is then incorporated in the computations as a modification of the eddy-viscosity, $\mu_t^\star = \gamma\mu_t$. For turbulence closure the SST model of Menter (1994) was used.

Important to note is that Suzen & Huang (2000) used a boundary layer code to solve the above equations. This approach is probably more convenient, especially when it comes to determining, for example, Re_{θ_t}, than solving the equations in an elliptic manner. With the former approach all properties are computed once at a certain streamwise location where after one proceeds with the next downstream position. In an elliptic solver all computational points are treated implicitly and the solution is found by iterating an initial guess to a fully converged solution. This means that the position of transition onset may vary during the iteration process, which possibly can cause numerical problems.

Four test cases were computed. They are the T3A and T3B (flat plate, ZPG) and the T3C1 and T3C2 (continuously changing pressure gradient, representing an aft-loaded turbine blade). The model works rather well for the T3A case but predicts too a late transition in the T3B case. The change in shape factor is in both cases, as expected due to the model's formulation, somewhat abrupt. The predicted skin friction coefficients agree well with experiments for the T3C1 and T3C2 cases and clearly outperform the Launder-Sharma model. Note that pressure coefficients from experiments were used as input for the boundary layer code computations.

Using essentially the same intermittency model as Suzen & Huang (2000), Suzen *et al.* (2002) computed a series of low-pressure turbines test cases in order to further validate the model. The only change to the intermittency equation is that the strain rate magnitude in Eqn 102, S, has been replaced with the magnitude of the vorticity vector, W.

As the low-pressure turbine flow is more complex than the T3 transition test cases predicting the location of transition onset becomes more difficult. Recall from above that transition onset was computed algebraically. Therefore Suzen *et al.* (2002) use different correlations for separated-flow and attached-flow transition. They used Eqn 82 and Eqn 79 for separated and attached flow transition, respectively.

In one of the computed cases predictions using the intermittency model are compared with the Menter SST and the Launder-Sharma models. Substantial improvements were found along the suction side wall as compared with the two “standard” turbulence models. Note that in the experiments only one passage between two walls shaped as a suction and a pressure side of a blade was measured and that some uncertainties in boundary condition specifications exists. The predictions are in general in better agreement with experiments in the higher freestream turbulence intensity cases.

5 Summary

This study is aimed at providing an overview of existing literature on the subject of transition modelling. It begins with a classification of three different modes of transition–natural, by-pass and separated flow induced transition. Then follows a short listing of certain flow features that are known to affect the transitional behavior. It is recognized that modelling natural transition is probably out of the scope of statistical descriptions, or even LES, and requires numerical procedures resolving all flow scales present. Indeed, one objective of the present study was to investigate the existing approaches to transition modelling and how they may be coupled to standard (RANS) statistical turbulence modelling within a typical multi-purpose CFD code framework(e.g. elliptic solver for arbitrary geometry and boundary conditions). Then follows a description of earlier ideas, all stemming from the classical work of Emmons (1951) on the formation of turbulent spots, leading to the intermittency distributions of Dhawan & Narasimha (1958) that are still commonly being used in both industry and academia. With the introduction of intermittency there was soon a need for a theoretical analogue to the experimental technique ‘conditional sampling’, i.e. a measure to treat, or model, turbulent and laminar flow portions separately. Conditional averages were introduced by Libby (1974) and it wasn’t long before full transport equations for the intermittency factor was derived. Discussed is also the so called low Reynolds number modelling of near wall turbulence *and* transition. Due to the fact that this method was originally suggested for turbulence modelling it is, in priciple, straightforward to use also to predict transitional phenomena and need no additional numerical treatment when nonturbulent flow regions are considered. The ease of using this approach is perhaps one of the reasons why the it has been so commonly studied. A complicating factor, however, are numerical issues like harsh requirements on grid fineness and problems with solutions not being unique.

Then follows a description of two more recent ideas. One, by Menter *et al.*

(2004) is based on the idea to incorporate experimental correlations in multi-purpose codes via a transport equation for a critical Reynolds number with a source term partly determined from a correlation based on experimental experiences. The second approach is the introduction of a transport equation for a laminar (nonturbulent) fluctuating energy of Walters & Leylek (2004). The origin of this idea is the presence of streamwise fluctuations in the pretransitional boundary layer, so called Klebanoff modes, that are precursors to turbulent spots in by-pass transition. The modelled build-up of the laminar kinetic energy eventually triggers the modelled production of turbulent fluctuations, a scenario that closely resembles that seen in numerical simulations.

Finally, in an effort to make this review less incomplete[5], several additional attempts to model transition were collected in Section 4. It is the authors' hope that this collection of earlier work will be a valuable starting point for further, more detailed, reading of the different approaches ranging from rather simple algebraic models to complex transport equations for the intermittency factor. Here is also where most of the referenced studies on separation induced transition are given. In the other sections focus is primarily on transition induced by freestream turbulence, i.e. by-pass transition.

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[5]As the subject of transition modelling is vast and the number of computational studies where different models have been investigated probably is even more overwhelming, including all relevant work would be an almost impossible mission.

A Correlation transport equation

Recently a new approach to transition modelling was suggested by Menter *et al.* (2004). The model uses the SST turbulence model for turbulence closure and solves two additional equations that describe the state of the boundary layer. The model was intended for use in a multi-purpose CFD code, which is why a requirement that only local fbw variables are allowed in its formulation was introduced in the development of the model. This is a highly desirable property unique of all transition models presented herein.

The proposed equations are transport equations for an intermittency facor and a “local transition onset momentum thickness Reynolds number”, $\tilde{Re}_{\theta t}$. The latter is based on an (arbitrary) experimental correlation, which may or may not include non-local quantities. The suggested correlation involves the local freestream velocity as the only non-local quantity (note that some features of the SST model also involves a wall distance dependence).

The intermittency factor is governed by:

$$\frac{\partial(\rho\gamma)}{\partial t} + \frac{\partial(\rho U_j \gamma)}{\partial x_j} = P_{\gamma 1} - E_{\gamma 1} + P_{\gamma 2} - E_{\gamma 2} + \frac{\partial}{\partial x_j}\left[\left(\mu + \frac{\mu_t}{\sigma_f}\right)\frac{\partial \gamma}{\partial x_j}\right] \quad (106)$$

where

$$\begin{aligned} P_{\gamma 1} &= F_{length}\rho S \left[\gamma F_{onset}\right]^{c_{a1}} && (107) \\ E_{\gamma 1} &= c_{e1} P_{\gamma 1}\gamma && (108) \end{aligned}$$

The function F_{onset} is used to trigger transition and is a function of the vorticity Reynolds number

$$\begin{aligned} F_{onset\ 1} &= \frac{Re_V}{2.193 Re_{\theta c}} && (109) \\ F_{onset\ 2} &= \min\left(\max\left(F_{onset\ 1}, F_{onset\ 1}^4\right), 2.0\right) && (110) \\ F_{onset\ 3} &= \max\left(1 - \left(\frac{R_T}{2.5}\right)^3,\ 0\right) && (111) \\ F_{onset} &= \max\left(F_{onset\ 2} - F_{onset\ 3},\ 0\right) && (112) \\ Re_V &= \frac{\rho y^2 S}{\mu} && (113) \\ Re_T &= \frac{\rho k}{\mu\omega} && (114) \\ Re_{\theta c} &= f(\tilde{Re}_{\theta t}) && (115) \end{aligned}$$

The functions $f(\tilde{Re}_{\theta t})$ and F_{length} are at the moment classified. The other source terms are computed as

$$\begin{aligned} P_{\gamma 2} &= c_{a2}\rho\Omega\gamma F_{turb} && (116) \\ E_{\gamma 2} &= c_{e2}P_{\gamma 2}\gamma && (117) \\ F_{turb} &= \exp\left[-\left(\frac{Re_T}{4}\right)^4\right] && (118) \end{aligned}$$

The constants used are:

$$c_{e1} = 1.0 \quad c_{a1} = 0.5 \quad c_{e2} = 50 \quad c_{a2} = 0.03 \quad \sigma_f = 1.0 \tag{119}$$

$Re_{\theta c}$ is the critical Reynolds number where the intermittency starts to grow in the boundary layer, F_{length} is used to control the length of the transition region whereas F_{turb} disables the destruction/relaminarization sources outside laminar boundary layers and in the viscous sublayer. At inlets the intermittency is set to unity and a zero-flux condition is specified at walls. First node y^+ values of about unity are required for the transition model to work properly.

The transport equation for the local transition onset momentum thickness Reynolds number", $\tilde{Re}_{\theta t}$, reads

$$\frac{\partial(\rho\tilde{Re}_{\theta t})}{\partial t} + \frac{\partial(\rho U_j\tilde{Re}_{\theta t})}{\partial x_j} = P_{\theta t} + \frac{\partial}{\partial x_j}\left[\sigma_{\theta t}\left(\mu+\mu_t\right)\frac{\partial\tilde{Re}_{\theta t}}{\partial x_j}\right] \tag{120}$$

The source term, through which the effect of the chosen onset correlation enters, is defined as

$$P_{\theta t} = c_{\theta t}\frac{\rho}{t}\left(Re_{\theta t} - \tilde{Re}_{\theta t}\right)(1.0 - F_{\theta t}) \tag{121}$$

where $t = 500\mu/(\rho U^2)$ is a time scale present for dimensional reasons. Another "blending" function, $F_{\theta t}$, turns off the source term in the boundary layer and allows $\tilde{Re}_{\theta t}$ to diffuse in from the freestream, its value ranging from zero in the freestream to unity in the boundary layer. It is computed as

$$F_{\theta t} = \min\left(\max\left(F_{wake}\exp\left[\left(-\frac{y}{\delta}\right)^4\right], 1.0 - \left(\frac{\gamma - \frac{1}{c_{e2}}}{1.0 - \frac{1}{c_{e2}}}\right)^2\right), 1.0\right) \tag{122}$$

where

$$\delta = \frac{50\Omega y}{U}\delta_{BL}, \quad \delta_{BL} = \frac{15}{2}\theta_{BL}, \quad \theta_{BL} = \frac{\tilde{Re}_{\theta t}\mu}{\rho U} \tag{123}$$

$$F_{wake} = \exp\left[\left(\frac{Re_\omega}{10^5}\right)^2\right], \quad Re_\omega = \frac{\rho\omega y^2}{\mu} \tag{124}$$

A zero flux wall boundary condition is used for $\tilde{R}e_{\theta t}$ and at inlets $\tilde{R}e_{\theta t}$ is computed from the empirical correlation. The final model constants are

$$c_{\theta t} = 0.03, \quad \sigma_{\theta t} = 10.0 \tag{125}$$

As the model was found to consistently overpredict the length of separated flow regions the following modification was suggested

$$\begin{aligned}
\gamma_{sep} &= \min\left(s_1 \max\left(\frac{Re_V}{2.193 Re_{\theta c}} - 1.0,\ 0.0\right) F_{reattach},\ 5\right) F_{\theta t} && (126)\\
F_{reattach} &= \exp\left[-\left(\frac{Re_T}{15}\right)^4\right] && (127)\\
\gamma_{eff} &= \max\left(\gamma,\ \gamma_{sep}\right) && (128)
\end{aligned}$$

This modification is argued to increase the production of turbulence kinetic energy by allowing γ to exceed unity. The model constant is given the value $s_1 = 8$. By modifying its value the size of the separation bubble can be controlled.

The transition model is coupled to the SST turbulence model via the production and destruction terms in the k equation, i.e.,

$$\begin{aligned}
\tilde{P}_k &= \gamma_{eff} P_k \\
\tilde{D}_k &= \min\left(\max\left(\gamma_{eff},\ 0.1\right),\ 1.0\right) D_k
\end{aligned} \tag{129}$$

where P_k and D_k are the original SST model production and destruction terms. Another change of the SST model is that the blending function had to be redefined. When the transition model is used it is computed as

$$\begin{aligned}
F_1 &= \max\left(F_1^{original},\ F_3\right) && (130)\\
F_3 &= \exp\left[-\left(\frac{R_y}{120}\right)^8\right] \\
R_y &= \frac{\rho y k^{1/2}}{\mu}
\end{aligned}$$

B The model of Walters & Leylek (2004)

The three governing transport equations read

$$\frac{Dk_T}{Dt} = P_T + R + R_{NAT} - \varepsilon - D_T + \frac{\partial}{\partial x_j}\left(\left(\nu + \frac{\alpha_T}{\sigma_k}\right)\frac{\partial k_T}{\partial x_j}\right) \quad (131)$$

$$\frac{Dk_L}{Dt} = P_L - R - R_{NAT} - D_L + \frac{\partial}{\partial x_j}\left(\nu\frac{\partial k_T}{\partial x_j}\right) \quad (132)$$

$$\begin{aligned}\frac{D\varepsilon}{Dt} &= C_{\varepsilon 1}\frac{\varepsilon}{k_T}\left(P_T + R_{NAT}\right) + C_{\varepsilon R}R\frac{\varepsilon}{\sqrt{k_T k_L}} - C_{\varepsilon 2}\frac{\varepsilon^2}{k_T} - \frac{\varepsilon}{k_T}D_T \\ &+ \frac{\partial}{\partial x_j}\left(\left(\nu + \frac{\alpha_T}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right)\end{aligned} \quad (133)$$

The production of turbulence by small scale turbulence, P_T, is computed as:

$$P_T = \nu_{T,s}S^2 \quad (134)$$

where

$$\nu_{T,s} = f_\mu f_{\tau,s} C_\mu \sqrt{k_{T,s}}\lambda_{eff} \quad (135)$$

A variable C_μ is used according to

$$C_\mu = \left(A_0 + A_s\left(\frac{Sk_T}{\varepsilon}\right)\right)^{-1} \quad (136)$$

in order to avoid stagnation point problems to some extent, whereas the damping functions f_μ and $f_{\tau,s}$ are given by

$$f_\mu = 1 - \exp\left(-\frac{\sqrt{Re_{T,s}}}{A_\nu}\right) \quad (137)$$

$$f_{\tau,s} = 1 - \exp\left(-C_{\tau,s}\left(\frac{\tau_m}{\tau_{T,s}}\right)^2\right) \quad (138)$$

where the small scale turbulence Reynolds number, the mean time scale and the effective turbulence time scales are given by:

$$Re_{T,s} = \frac{k_{T,s}^2}{\nu\varepsilon}, \quad \tau_m = \frac{1}{S}, \quad \tau_{T,s} = \frac{\lambda_{eff}}{\sqrt{k_{T,s}}} \quad (139)$$

Analogous to the turbulent production the production of laminar fluctuations is computed as

$$P_L = \nu_{T,l}S^2 \quad (140)$$

where

$$\nu_{T,l} = f_{\tau,l} C_{1l} \left(\frac{\Omega \lambda_{\mathit{eff}}^2}{\nu} \right) \sqrt{k_{T,l}} \lambda_{\mathit{eff}} \tag{141}$$

with damping function and effective time scale given by

$$f_{\tau,l} = 1 - \exp\left(-C_{\tau,l} \left(\frac{\tau_m}{\tau_{T,l}} \right)^2 \right) \tag{142}$$

$$\tau_{T,l} = \frac{\lambda_{\mathit{eff}}}{\sqrt{t_{T,l}}} \tag{143}$$

The near wall dissipation rates, D_T and D_L, are computed as

$$D_T = 2\nu \frac{\partial \sqrt{k_T}}{\partial x_j} \frac{\partial \sqrt{k_T}}{\partial x_j} \tag{144}$$

$$D_L = 2\nu \frac{\partial \sqrt{k_L}}{\partial x_j} \frac{\partial \sqrt{k_L}}{\partial x_j} \tag{145}$$

and the total dissipation is

$$\varepsilon_{TOT} = \varepsilon + D_T + D_L \tag{146}$$

The term R represent the coupling between the laminar fluctuations and 'real' turbulence. It is the rate of conversion of fluctuating laminar kinetic energy to turbulence kinetic energy, hence a sink in the LKE equation and a source in the TKE equation. It is modelled as

$$R = C_R \beta_{BP} \frac{k_L}{\tau_T}, \quad \tau_T = \frac{\lambda_{\mathit{eff}}}{\sqrt{k_T}} \tag{147}$$

where β_{BP} is another damping function, expressed in a wall distance measure (usually denoted R_y), used to control the by-pass transition process.

$$\beta_{BP} = 1 - \exp\left(-\frac{\phi_B P}{A_{BP}} \right) \tag{148}$$

$$\phi_{BP} = \max\left(\frac{\sqrt{k_T} d}{\nu} - C_{BP,crit}\,, 0 \right) \tag{149}$$

To rapidly reduce the turbulence length scale in the early stages of transition C_R was given the form

$$C_R = 0.21 \left(\frac{1.5 \lambda_T}{\lambda_{\mathit{eff}}} - 1 \right) \tag{150}$$

$A_0 = 4.04$	$C_{BP,crit} = 35$	$C_{\tau,l} = 4360$
$A_S = 2.12$	$C_{1l} = 3.4 \times 10^{-6}$	$C_{\tau,s} = 4360$
$A_\nu = 5.5$	$C_\lambda = 2.495$	$C_{\varepsilon,2} = 1.92$
$A_{BP} = 8$	$C_R = 0.21$	$\sigma_k = 1$
$\sigma_\varepsilon = 1.4$	$C_{\mu,std} = 0.09$	

Table 1: Model constants of the turbulence/transition model of Walters & Leylek (2004).

The ε equation coefficient $C_{\varepsilon 1}$ is used to decrease the turbulence length scale in the vicinity of wall by assigning it the following form

$$C_{\varepsilon 1} = 2\left(1 - \left(\frac{\lambda_{eff}}{\lambda_T}\right)^{4/3}\right) + 1.44\left(\frac{\lambda_{eff}}{\lambda_T}\right)^{4/3} \tag{151}$$

The total eddy-viscosity, eddy diffusivity and turbulent scalar diffusivity are computed as

$$\nu_{TOT} = \nu_{T,s} + \nu_{T,l} \tag{152}$$

$$\alpha_{\theta,TOT} = \frac{\nu_{T,s}}{Pr_\theta} + C_\theta\sqrt{k_{T,l}}\lambda_{eff} \tag{153}$$

$$\alpha_T = f_\mu C_{\mu,std}\sqrt{k_{T,s}}\lambda_{eff} \tag{154}$$

A zero flux boundary condition is used at the wall for all three equations. All model constants are given in Table 1. They were tuned in a fully developed channel flow and a transitional flat plate flow simulation.

Note that the original model also contain features that allow it to model natural and mixed natural/by-pass transition, i.e. the modelling of the term R_{NAT} (cf. Eqn 133-133). For details see Walters & Leylek (2004).

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