CHALMERS

Internal Report 04/02

Anisotropic Formulation of the Velocity Correlation Tensor

Mattias Billson, Lars-Erik Eriksson, Peter Jordan and Lars Davidson

Division of Thermo and Fluid Dynamics
Department of Mechanical Engineering
CHALMERS UNIVERSITY OF TECHNOLOGY
Göteborg, Sweden, May 2004

Anisotropic Formulation of the Velocity Correlation Tensor

Mattias Billson*, Lars-Erik Eriksson*,
Peter Jordan† and Lars Davidson*

* Div. of Thermo and Fluid Dynamics
Dept. of Mechanical Engineering
Chalmers University of Technology
SE-412 96 Gẗeborg, Sweden

† Laboratoire d'Etudes Aérodynamiques,
UMR CNRS 6609, Université de Poitiers, France

Contents

1 Introduction

The aim of the present report is to present a new statistical model for turbulence which may be used in classical Lighthill's analogy based jet noise theory. The focus will be on the spatial two-point velocity correlation tensor $R_{ij}(\mathbf{r})$ (defined below) which is central in all statistical description of turbulence and of special importance in aeroacoustic theory.

Statistical models for turbulence-generated noise have been developed ever since 1952 when Sir James Lighthill presented the first of two papers on aerodynamically generated sound [1, 2]. In these models, the two-point velocity correlation tensor is the basis of the statistical representation of the turbulence structure and is strongly related to the sound emission directivity.

A model [3] based on Lighthill's analogy for the sound generation from unsteady flow and the relation to $R_{ij}(\mathbf{r})$ will be described first. Next, two existing theories (isotropic [4] and axisymmetric [5] turbulence) which have been used to model the two-point velocity correlation tensor $R_{ij}(\mathbf{r})$ are presented. The validity of these theories in the sound generating region of a jet is then assessed using data from an LES simulation of a high Mach number jet [6]. The basis of a more general model of $R_{ij}(\mathbf{r})$ for anisotropic homogeneous turbulence is then derived and preliminary results are presented. It is however acknowledged that more theoretical work is needed before a closed model of anisotropic homogeneous turbulence can be presented.

2 The sound power in terms of turbulence statistics

Based on Lighthill's acoustic analogy [1, 2], Ribner [3] presented an expression for the sound power emitted from localized unsteady flow as

$$
\begin{aligned}
P(\mathbf{x}) &= \int_{\infty} P(\mathbf{x},\mathbf{y})\mathrm{d}^3\mathbf{y} \\
P(\mathbf{x},\mathbf{y}) &= A\frac{x_i x_j x_k x_l}{x^4}\int_{\infty}\frac{\partial^4}{\partial\tau^4}R_{ijkl}(\mathbf{y},\mathbf{r},\tau)\mathrm{d}^3\mathbf{r} \\
R_{ijkl}(\mathbf{y},\mathbf{r},\tau) &= \overline{v_i v_j v_k' v_l'} \\
v_i &= v_i(\mathbf{y},t) \quad v_k' = v_k(\mathbf{y}+\mathbf{r},t+\tau)
\end{aligned}
\tag{1}
$$

where $A = \rho_0/(16\pi^2 c_0^5 x^2 C^5)$, ρ_0 is the ambient air density, c_0 the ambient speed of sound, C is the convective amplification factor $(1 - M_c\cos\theta)$ and θ is the angle between the mean flow direction and the direction to the observer. The assumptions implicit in the above formulation is that the observer location $\mathbf{x}$ is far away from the source location $\mathbf{y}$ and that the length scale of the source is small compared to the wave length of the emitted sound. The Reynolds number of the turbulence is furthermore assumed to be high and entropy fluctuations small such that only $\rho v_i v_j$ is retained in the Lighthill source term.

By decomposing the velocities into a mean and fluctuating part as

$$v_i = U\delta_{i1} + u_i \tag{2}$$

where the mean flow U is assumed to be in the y_1 direction and u_i is the fluctuating velocity component, the fourth-order velocity correlations in equation 1 can be written as

$$\begin{aligned}
R_{1111} &= \overline{u_1^2 u_1'^2} + 4UU'\overline{u_1 u_1'} \\
R_{2222} &= \overline{u_2^2 u_2'^2} \\
R_{3333} &= \overline{u_3^2 u_3'^2} \\
R_{1212,2121,1221,2112} &= \overline{u_1 u_2 u_1' u_2'} + UU'\overline{u_2 u_2'} \\
R_{1313,3131,1331,3113} &= \overline{u_1 u_3 u_1' u_3'} + UU'\overline{u_3 u_3'} \\
R_{2323,3232,2332,3223} &= \overline{u_2 u_3 u_2' u_3'} \\
R_{1122,2211} &= \overline{u_1^2 u_2'^2} \\
R_{1133,3311} &= \overline{u_1^2 u_3'^2} \\
R_{2233,3322} &= \overline{u_2^2 u_3'^2}
\end{aligned} \tag{3}$$

where the notation R_{ijkl} is short for $R_{ijkl}(\mathbf{y}, \mathbf{r}, \tau)$. The correlations involving fluctuations only and combinations of fluctuations and the mean flow in equation 3 were by Ribner denoted by self-noise and shear-noise contributions respectively. The shear-noise terms involve second order correlations of velocity fluctuations whereas the self-noise terms involve fourth-order correlations of velocity fluctuations. All other combinations of velocity correlations as a result of the decomposition of the velocities in equation 2 are shown by Goldstein and Rosenbaum [7] to be zero for homogeneous turbulence.

Under the assumption of joint normal probability distribution between velocity components, the fourth-order velocity correlation can be written as [4]

$$\overline{u_i u_j u_k' u_l'} = \overline{u_i u_j}\,\overline{u_k' u_l'} + \overline{u_i u_k'}\,\overline{u_j u_l'} + \overline{u_i u_l'}\,\overline{u_j u_k'} \tag{4}$$

but upon noting that $\overline{u_i u_j}\,\overline{u_k' u_l'}$ is independent of time separation τ this term will vanish when evaluating equation 1 and can therefore be omitted in the analysis. The correlations $\overline{u_i u_j'}$ are postulated to be factorable into a space factor and a time factor as

$$\overline{u_i u_j'} \equiv R_{ij}(\mathbf{y}, \mathbf{r}, \tau) = R_{ij}(\mathbf{y}, \mathbf{r}) g(\tau) \tag{5}$$

where $R_{ij}(\mathbf{y}, \mathbf{r}) = R_{ij}(\mathbf{y}, \mathbf{r}, \tau = 0)$ is the two-point velocity correlation tensor with zero time separation between the points $(\cdot)$ and $(\cdot)'$. The time factor $g(\tau)$ will for now be unspecified but a number of different functions have been suggested [3, 8–10].

The emission directivity $\mathrm{dir}(ijkl) \equiv (x_i x_j x_k x_l)/(x^4)$ of different terms $R_{ijkl}(\mathbf{y}, \mathbf{r}, \tau)$ can in polar coordinates (x, ϕ, θ) defined by (see figure 1)

$$\begin{aligned}
x_1 &= x\cos\theta \\
x_2 &= x\sin\theta\cos\phi \\
x_3 &= x\sin\theta\sin\phi
\end{aligned} \tag{6}$$

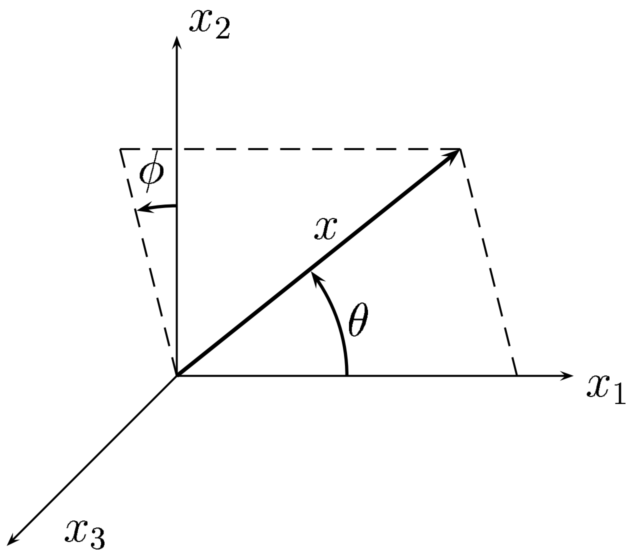


Figure 1: Orientation of spherical coordinate system for directivities

be expressed as

$$\begin{aligned}
\mathrm{dir}(1111) &= \cos^4\theta \\
\mathrm{dir}(2222) &= \sin^4\theta\cos^4\phi \\
\mathrm{dir}(3333) &= \sin^4\theta\sin^4\phi \\
\mathrm{dir}(1212) &= \cos^2\theta\sin^2\theta\cos^2\phi \\
\mathrm{dir}(1313) &= \cos^2\theta\sin^2\theta\sin^2\phi \\
\mathrm{dir}(2323) &= \sin^4\theta\cos^2\phi\sin^2\phi
\end{aligned} \tag{7}$$

where the indices are interchangeable, for example $\mathrm{dir}(1212) = \mathrm{dir}(1122)$ as long as the values of the indices are not changed. The directivities in equation 7 show that different correlations $R_{ijkl}(\mathbf{y}, \mathbf{r}, \tau)$ generate sound which will have different directivity patterns. The specific model of $R_{ij}(\mathbf{y}, \mathbf{r})$ which gives $R_{ijkl}(\mathbf{y}, \mathbf{r}, \tau)$ through equations 4 and 5 will thus have an impact on the directivity of the emitted sound.

For the purpose of jet noise predictions, the two-point velocity correlation tensor $R_{ij}(\mathbf{y}, \mathbf{r})$ has in previous presentations been modeled based on isotropic [3, 9] turbulence theory by Batchelor [4] as well as based on axisymmetric [7, 8, 11–13] turbulence theory by Chandrasekhar [5]. The impact of choosing different models for $R_{ij}(\mathbf{y}, \mathbf{r})$ will be addressed in the following sections and a new more general anisotropic model will be proposed.

3 Existing models of the two-point velocity correlation tensor

There are two main turbulence theories which have been used to express a functional form for $R_{ij}(\mathbf{y}, \mathbf{r})$. These are shown in the table below together with references of different presentations where they have been applied. The models for $R_{ij}(\mathbf{y}, \mathbf{r})$ based on these theories (isotropic and axisymmetric) will briefly be presented in the following subsections. See the references for more details.

- Isotropic turbulence (Batchelor) [4]
  - Ribner [3]
  - Bailly *et al.* [9]
- Axisymmetric homogeneous turbulence (Chandrasekhar) [5]
  - Goldstein and Rosenbaum [7]
  - Khavaran [8]
  - Bechara *et al.* [11]
  - Devenport *et al.* [12]
  - Jordan and Gervais [13]
  - Bailly *et al.* [9]

3.1 Isotropic model by Batchelor

For isotropic turbulence, Batchelor [4] presented a simple form of the structure of $R_{ij}(\mathbf{r}) = R_{ij}(\mathbf{y}, \mathbf{r})$ in terms of the longitudinal correlation function $f(r)$. The derivation of the relation was based on using the symmetries and direction independence of isotropic turbulence and is presented in Batchelor [4]. Isotropic turbulence is also homogeneous so there is no $\mathbf{y}$-dependence in $R_{ij}(\mathbf{r})$. The final expression reads

$$R_{ij}^{iso}(\mathbf{r}) = \overline{u^2}\left(\frac{f(r) - g(r)}{r^2} r_i r_j + g(r)\delta_{ij}\right) \tag{8}$$

where $f(r)$ and $g(r)$ are the longitudinal and transversal correlation functions respectively. The transversal correlation function $g(r)$ is for isotropic turbulence related to $f(r)$ as

$$g(r) = f(r) + \frac{r}{2}\frac{\partial f(r)}{\partial r} \tag{9}$$

The amplitude $\overline{u^2}$ is the mean-square of any velocity component since for isotropic turbulence they are all the same. The length scales related to $\mathbf{R}_{ij}^{iso}(\mathbf{r})$ are isotropic and controlled by the longitudinal correlation function $f(r)$ which is the same for all directions.

3.1.1 Model by Ribner

Ribner [3] used the isotropic model of the velocity correlation $\mathbf{R}_{ij}^{iso}(\mathbf{r})$ by Batchelor to model the source term in equation 1 for the purpose of jet noise predictions. What was left to model before the isotropic velocity correlation tensor could be evaluated was the functional form of $f(r)$. Ribner used a Gaussian shape of $f(r)$ as

$$f = e^{-\pi r^2/L^2}; \quad r^2 = r_1^2 + r_2^2 + r_3^2 \tag{10}$$

following Lilley [14]. The length scale L in equation 10 is the longitudinal length scale and the amplitude $\overline{u^2}$ in equation 8 was chosen by Ribner to be computed from the axial velocity component in the jet, i.e. $\overline{u^2} = \overline{u_1^2}$.

The assumption of local isotropy in the model of the correlation tensor is not very accurate in the interesting regions of a jet. It is well known that the energy of the different velocity components differ quite a lot with u_1-component being the largest one for the jet mixing layer and transition region [6]. The length scales are also highly anisotropic in the inhomogeneous regions of a jet.

The effect of using an isotropic model of the correlation tensor is that the self-noise part of the emitted sound will have a uniform directivity (if the effect of convection is neglected) which would not be the case if an anisotropic model would be used for $\mathbf{R}_{ij}(\mathbf{r})$.

3.2 Axisymmetric model by Chandrasekhar

To be able to introduce some degree of anisotropy in the model, Goldstein and Rosenbaum [7] used the theory of axisymmetric homogeneous turbulence developed by Chandrasekhar [5] to model the correlation tensor $R_{ij}(\mathbf{y}, \mathbf{r})$. The theory of axisymmetric homogeneous turbulence was derived for turbulence which is isotropic in a plane perpendicular to a direction vector $\boldsymbol{\lambda}$ but anisotropic with respect to directions out of the plane. Similar models based on axisymmetric turbulence have been presented by Khavaran [8], Devenport *et al.* [12] and Jordan and Gervais [13].

Chandrasekhar [5] presented in 1950 a complete theory governing axisymmetric turbulence including the determination of the functional form of the two-point correlation tensor $\mathbf{R}_{ij}(\mathbf{r})$ in terms of a set of scalar functions. The derivation is quite extensive and only the final expressions are given here.

In a coordinate system $\tilde{\mathbf{y}}$ with separation vector $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$ where index 1 is a direction parallel to an axis of symmetry $\boldsymbol{\lambda}$ and (u_1, u_2, u_3) are the corresponding velocity components, the following holds for homogeneous axisymmetric turbulence.

$$\begin{aligned} \overline{u_2^2} &= \overline{u_3^2} \\ \overline{u_1^2} &= 2(k - \overline{u_2^2}) \end{aligned} \tag{11}$$

where $\overline{u_i^2}$ are the normal components of the Reynolds stress tensor and k is the turbulence kinetic energy. A more general statement related to axisymmetric turbulence is that "the mean value of any function of the [fluctuating] velocities and their derivatives be invariant, ... for rotations about a preferred direction $\boldsymbol{\lambda}$" [5].

This means that the velocity correlation tensor $\mathbf{R}_{ij}(\mathbf{r})$ also experiences this axisymmetry.

Under the assumption of homogeneous incompressible axisymmetric turbulence the two-point velocity correlations can be expressed as

$$R_{ij}^{axi}(\tilde{\mathbf{y}}, \boldsymbol{\xi}) = \overline{u_i u'_j} = \epsilon_{jlm} \frac{\partial q_{im}}{\partial \xi_l} \tag{12}$$

where $u_i = u_i(\tilde{\mathbf{y}}, t)$ and $u'_j = u_j(\tilde{\mathbf{y}} + \boldsymbol{\xi}, t + \tau)$. The skew tensor q_{im} in equation 12 is defined by

$$q_{im} = \xi_k[\epsilon_{imk}Q_1 + \epsilon_{i1k}(\delta_{1m}Q_2 + \xi_m Q_3)] \tag{13}$$

where scalar functions Q_1, Q_2 and Q_3 are functions of the separation distance ξ and the direction $\boldsymbol{\xi}/\xi$. It is possible to reduce the number of independent scalar functions from three to two by using the identity $\overline{u_i u'_j} = \overline{u_j u'_i}$ in equations 12 and 13. This leads to the following relations between Q_1, Q_2 and Q_3

$$\left(\xi_3 \frac{\partial}{\partial \xi_2} - \xi_2 \frac{\partial}{\partial \xi_3} \right) Q_i = 0 \quad ; \qquad i = 1, 2 \tag{14}$$

and

$$Q_3 = \left(\frac{\partial}{\partial \xi_1} - \frac{\xi_1}{\xi_3} \frac{\partial}{\partial \xi_3} \right) Q_1 \tag{15}$$

Equation 15 is used to reduce the problem into having to specify only two scalar functions Q_1 and Q_2. The resulting velocity correlations (equation 12) are axisymmetric in the y_1-direction with the following single point ($\xi = 0$) properties

$$\begin{aligned} \overline{u_2 u_2} &= \overline{u_3 u_3} \\ \overline{u_1 u_1} &\neq \overline{u_2 u_2} \\ \overline{u_1 u_1} &\neq \overline{u_3 u_3} \\ \overline{u_1 u_2} &= \overline{u_1 u_3} \\ \overline{u_2 u_3} &= 0 \end{aligned} \tag{16}$$

where $\neq$ denotes "not necessarily equal to".

3.2.1 Model by Goldstein and Rosenbaum

Goldstein and Rosenbaum [7] suggested the following exponential forms of the two scalar functions Q_1 and Q_2 compatible with equation 14 (the time separation dependence in the original formulation has been omitted in the present form)

$$\begin{aligned} Q_1(\tilde{\mathbf{y}}, \xi_{23}, \xi_1) &= -\frac{1}{2}\overline{u_1^2}\exp\left[-\left[\left(\frac{\xi_{23}}{L_2}\right)^2 + \left(\frac{\xi_1}{L_1}\right)^2 \right]^{(1/2)} \right] \\ Q_2(\tilde{\mathbf{y}}, \xi_{23}, \xi_1) &= -(\overline{u_2^2} - \overline{u_1^2})\exp\left[-\left[\left(\frac{\xi_{23}}{L_2}\right)^2 + \left(\frac{\xi_1}{L_1}\right)^2 \right]^{(1/2)} \right] \end{aligned} \tag{17}$$

where $\xi_{23} = (\xi_2^2 + \xi_3^2)^{1/2}$ and $\tilde{\mathbf{y}}$ is the source location. The resulting turbulence has a certain length scale in the $\boldsymbol{\lambda}$-direction L_1 and a different length scale in the directions normal to $\boldsymbol{\lambda}$, i.e. $L_2 = L_3$. The parameters left to specify in equation 17 are the length scale L_1 and normal stresses $\overline{u_1^2}$ as well as the ratios L_1/L_2 and $\overline{u_1^2}/\overline{u_2^2}$ determining the anisotropy of the axisymmetric turbulence.

The orientation of the axisymmetry $\boldsymbol{\lambda}$ is arbitrary and two special cases were suggested by Goldstein and Rosenbaum to be of special interest for the round jet. These were the cases where $\boldsymbol{\lambda}$ was oriented in the axial and the radial directions of the jet respectively.

Radial direction

Goldstein and Rosenbaum considered the use of the radial direction as the direction of turbulence axisymmetry based on measurements in the shear layer of a jet by Bradshaw *et al.* [15]. The measurements were performed at the center of the shear layer at two diameters downstream of the jet nozzle exit. At this location the longitudinal correlation function and turbulence intensity in the direction of the flow were equal to those in the circumferential direction but differed from those in the radial direction. This supported the choice of the radial direction as the axis of symmetry $\boldsymbol{\lambda}$ (based on length scales and intensities). The resulting self-noise directivity from a unit volume presented in [7] showed 2.5 dB larger intensity levels in the axial direction than in the radial direction.

Axial direction

Goldstein and Rosenbaum also considered the use of the axial direction as the direction of turbulence axisymmetry. In the data from [15] further downstream (four diameters from the jet nozzle exit) the relation between the turbulence intensities in the radial and circumferential direction were equal but significantly smaller than that in the axial direction. If this direction (axial) was chosen as the direction of turbulence axisymmetry (based on turbulence intensities), the difference in self-noise directivity was 6 dB between the axial and radial directions.

3.2.2 Model by Khavaran

Khavaran [8] presented a modified model of the axisymmetric turbulence and applied it to a jet flow. The same axisymmetric theory was used for the functional form of the velocity correlation tensor $R_{ij}^{axi}(\tilde{\mathbf{y}}, \boldsymbol{\xi})$ but Khavaran suggested the following two Gaussian forms of the scalar functions Q_1 and Q_2 compatible with equation 14 (the time separation dependence in the original formulation has been omitted in the present form).

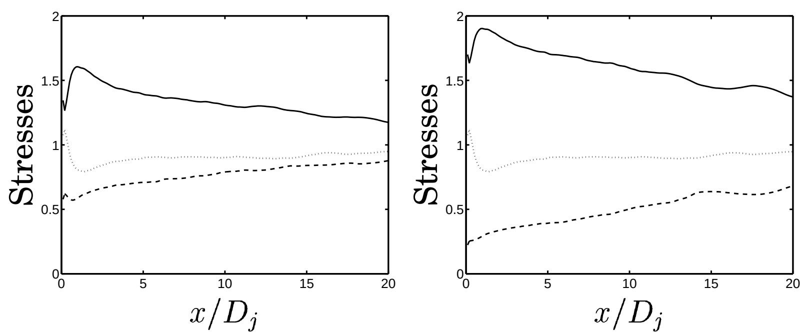
$$\begin{aligned} Q_1(\tilde{\mathbf{y}}, \xi_{23}, \xi_1) &= -\frac{1}{2}\overline{u_1^2}\exp\left[-\pi\left[\left(\frac{\xi_{23}}{L_2}\right)^2 + \left(\frac{\xi_1}{L_1}\right)^2\right]\right] \\ Q_2(\tilde{\mathbf{y}}, \xi_{23}, \xi_1) &= -(\overline{u_2^2} - \overline{u_1^2})\exp\left[-\pi\left[\left(\frac{\xi_{23}}{L_2}\right)^2 + \left(\frac{\xi_1}{L_1}\right)^2\right]\right] \end{aligned} \tag{18}$$

In Khavaran [8] only the axial direction was chosen as the axis of symmetry but the ratios L_1/L_2 and u_1^2/u_2^2 in the model were varied and the effect on the emitted sound was investigated. It was found that the self-noise directivity was sensitive to changes in the ratios, see Khavaran [8] for more details.

4 Considerations for real jets

In the applications above, the axisymmetric model of the velocity correlation tensor has been used in order to introduce turbulence anisotropy in Lighthill's analogy when applied to turbulent jets. The model is an improvement compared to the isotropic model of the velocity correlation tensor. The axisymmetric turbulence model is however not representative of turbulence in the sound producing regions of a jet such as the shear and transition regions. This is demonstrated by the plots in figures 2 and 3 where the Reynolds stresses in the middle of the shear layer of a Mach 0.75 are plotted for different axial positions. The stresses are sampled from an LES by Andersson *et al.* [6].

If the axial direction is chosen as the axis of symmetry $\boldsymbol{\lambda}$ in equations 12 and 13 the single point statistics in equation 16 are required to be approximately true. These stress terms evaluated from the LES are plotted in figure 2(a) and 3. Figure 2(a) shows that the normal stresses $\overline{u_2u_2}$ and $\overline{u_3u_3}$ are quite similar for all positions and $\overline{u_1u_1}$ is considerably larger. This is in accordance with axisymmetric turbulence. The cross-stresses $\overline{u_1u_2}$ and $\overline{u_1u_3}$ in figure 3 are however not of similar magnitude which is required by equation 16. The $\overline{u_1u_2}$ correlation is about the same magnitude of $\overline{u_2u_2}$ and should thus contribute to the sound directivity whereas $\overline{u_1u_3}$ is almost zero and should not. These correlations would though in an axisymmetric turbulence model oriented in the jet axial direction be equal and thus contributing equally to the generated sound. The axial direction is thus not suitable as an axis of turbulence symmetry and it is obvious from these arguments that a radial direction cannot be chosen as the axis of turbulence symmetry either.



(a) Expressed in ordinary coordinate system

(b) Expressed in principal coordinate system

Figure 2: Normalized normal Reynolds stress tensor components in the xy-plane in the shear-layer (normalized by $3/(2k)$). Solid line: $\overline{u_1u_1}$; dashed line: $\overline{u_2u_2}$; dotted line: $\overline{u_3u_3}$.

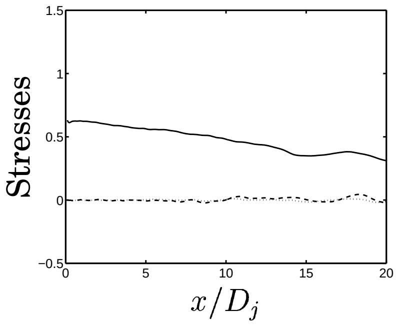


Figure 3: Normalized cross Reynolds stress tensor components in the xy-plane in the shear-layer (normalized by $3/(2k)$). Solid line: $\overline{u_1u_2}$; dashed line: $\overline{u_1u_3}$; dotted line: $\overline{u_2u_3}$.

A third direction could be attempted to be used as the axis of symmetry for the turbulence, namely a direction along one of the principal axes of the Reynolds stress tensor. In this coordinate system, all cross-stresses are zero which is valid in axisymmetric turbulence. No pair of the normal stresses are of equal magnitude however, see figure 2(b), so none of the principal axes can be used as axis of symmetry. Since the axisymmetric model does not accurately represent the turbulence in the shear layer of a jet, the isotropic model will not either.

5 Derivation of a more general model of the two-point velocity correlation tensor

The sections above demonstrate the need for a more general model of the two-point velocity correlation tensor $R_{ij}(\mathbf{y}, \mathbf{r})$. Both the isotropic and axisymmetric turbulence theories above are complete and govern exactly the functional form of $R_{ij}(\mathbf{y}, \mathbf{r})$ under the respective assumptions. When applied in equation 1 in order to perform a jet noise prediction however, there are complexities in the flow which are not valid in the assumptions of these theories.

Both theories include the assumption of homogeneous flow which, of course, is not the case in a jet flow. This assumption is though very hard to avoid in the derivations since quantities such as correlations and length scales are difficult to define unambiguously in general inhomogeneous flows.

As demonstrated in the section above the turbulence in a jet is neither isotropic nor axisymmetric. The choice of local length scales and turbulence stresses are thus somewhat arbitrary when these models are used to model $R_{ij}(\mathbf{y}, \mathbf{r})$. A more general model of $R_{ij}(\mathbf{y}, \mathbf{r})$ where anisotropy in turbulence intensities and length scales are part of the basis of the model is necessary to avoid these deficiencies.

An attempt to construct a more general model of $R_{ij}(\mathbf{y}, \mathbf{r})$ based on the homogeneous turbulence assumption but including anisotropy in turbulence intensities and length scales will be presented. A functional form of $R_{ij}(\mathbf{y}, \mathbf{r})$ in terms of six one-dimensional correlation functions is achieved. A closure model where these correlation functions are related to single point statistics is also presented. This closure is however preliminary and is not to be taken as a ready-to-use model.

The strategy is to express the second order two-point velocity correlation tensor $R_{ij}(\mathbf{r})$ in terms of a correlation $\widetilde{R}_{ij}(\widetilde{r}_1)$ where $\widetilde{r}_1$ denotes a separation in a coordinate system $\widetilde{\mathbf{x}}$ which has $\widetilde{x}_1$ parallel to $\mathbf{r}$. Define two coordinate systems $\mathbf{x} = (x_1, x_2, x_3)$ and $\widetilde{\mathbf{x}} = (\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3)$ with the same origin, see figure 4. Let $\widetilde{\mathbf{x}}$ be rotated compared to $\mathbf{x}$ such that $\widetilde{x}_1$ is parallel to a separation $\mathbf{r} = (r_1, r_2, r_3)$ expressed in the $\mathbf{x}$ coordinate system. One possible orthonormal set of rotation vectors relating $\mathbf{x}$ to $\widetilde{\mathbf{x}}$ is then

$$\begin{aligned} \mathbf{e_1} &= \left(\frac{r_1}{r}, \frac{r_2}{r}, \frac{r_3}{r}\right)^T \\ \mathbf{e_2} &= \left(\frac{-r_2}{r_h}, \frac{r_1}{r_h}, 0\right)^T \\ \mathbf{e_3} &= \left(\frac{r_1 r_3}{r r_h}, \frac{r_2 r_3}{r r_h}, -\frac{r_h^2}{r r_h}\right)^T \end{aligned} \tag{19}$$

where $r = (r_1^2 + r_2^2 + r_3^2)^{(1/2)}$ and $r_h = (r_1^2 + r_2^2)^{(1/2)}$ respectively. A rotation matrix $\mathbf{E} = (\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3})$ will relate $\mathbf{x}$ to $\widetilde{\mathbf{x}}$ and vise versa as

$$\begin{aligned} \mathbf{x} &= \mathbf{E}\,\widetilde{\mathbf{x}} \\ \widetilde{\mathbf{x}} &= \mathbf{E}^T\,\mathbf{x} \end{aligned} \tag{20}$$

and the spatial separation $\widetilde{\mathbf{r}} = (\widetilde{r}_1, \widetilde{r}_2, \widetilde{r}_3)$ will be related to $\mathbf{r}$ as

$$\widetilde{\mathbf{r}} = \mathbf{E}^T\,\mathbf{r} = (\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3})^T \mathbf{r} = (r, 0, 0)^T \tag{21}$$

i.e. any separation $\mathbf{r}$ results in a separation in the first component $\widetilde{r}_1$ in the coordinate system $\widetilde{\mathbf{x}}$. For simplicity the notation will here after be changed to tensor notation. The rotation matrix will be denoted by $E_{ij} = \mathbf{E}$.

With the assumption of homogeneous turbulence a velocity correlation tensor can be defined in the $\mathbf{x}$ coordinate system as

$$R_{ij}(\mathbf{r}) = \overline{u_i(\mathbf{x}) u_j(\mathbf{x} + \mathbf{r})} \tag{22}$$

where the overline $\overline{(\cdot)}$ denotes an appropriate average and u_i, u_j are velocity components. For homogeneous turbulence there is no dependence of $\mathbf{x}$ for $R_{ij}(\mathbf{r})$. By using the rotation matrix E_{ij}, the correlation tensor $R_{ij}(\mathbf{r})$ can then be expressed as

$$R_{ij}(\mathbf{r}) = E_{ik}\, E_{jl}\, \widetilde{R}_{kl}(\widetilde{\mathbf{r}}) = E_{ik}\, E_{jl}\, \widetilde{R}_{kl}(r, 0, 0) \tag{23}$$

where $\widetilde{R}_{ij}(r, 0, 0)$ are correlations of pairs of velocity components $\widetilde{u}$, $\widetilde{v}$ and $\widetilde{w}$ with separation in the $\widetilde{r}_1$-direction. These can be expressed individually as

$$\begin{aligned} \widetilde{R}_{11}(r, 0, 0) &= -\widetilde{\boldsymbol{\tau}}_{11} \widetilde{f}(r) \\ \widetilde{R}_{22}(r, 0, 0) &= -\widetilde{\boldsymbol{\tau}}_{22} \widetilde{g}(r) \\ \widetilde{R}_{33}(r, 0, 0) &= -\widetilde{\boldsymbol{\tau}}_{33} \widetilde{h}(r) \\ \widetilde{R}_{12}(r, 0, 0) &= -\widetilde{\boldsymbol{\tau}}_{12} \widetilde{f}_{12}(r) \\ \widetilde{R}_{13}(r, 0, 0) &= -\widetilde{\boldsymbol{\tau}}_{13} \widetilde{f}_{13}(r) \\ \widetilde{R}_{23}(r, 0, 0) &= -\widetilde{\boldsymbol{\tau}}_{23} \widetilde{f}_{23}(r) \end{aligned} \tag{24}$$

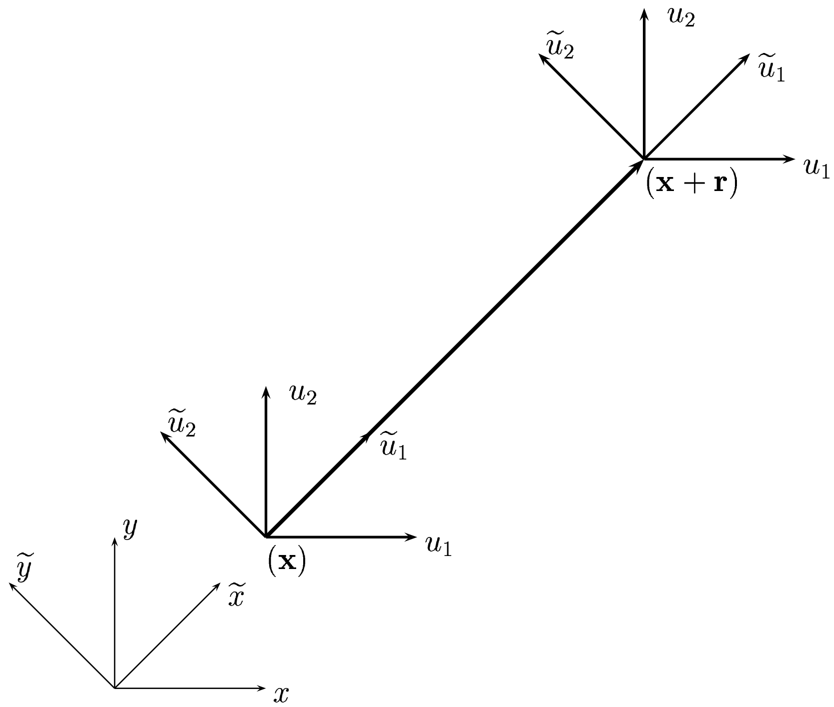


Figure 4: Original and rotated coordinate systems x and $\widetilde{\text{x}}$. Velocity components u_i and $\widetilde{u}_j$ in respective coordinate system.

where $\widetilde{f}(r)$ is the longitudinal velocity correlation function and $\widetilde{g}(r)$ and $\widetilde{h}(r)$ are the transversal correlation functions related to $\widetilde{v}$ and $\widetilde{w}$ respectively. $\widetilde{f}_{12}(r)$, $\widetilde{f}_{13}(r)$ and $\widetilde{f}_{23}(r)$ are the corresponding cross correlation functions. The assumption of homogeneous turbulence also ensures that $\widetilde{R}_{ij}(r,0,0) = \widetilde{R}_{ji}(r,0,0)$. The normalization is done by the correlation tensor at zero separation, i.e. the Reynolds stress tensor $\widetilde{\boldsymbol{\tau}}_{ij}$ which is computed from the stress tensor in the original coordinate system x as

$$\widetilde{\boldsymbol{\tau}}_{ij} = E_{ki}\, E_{lj}\, \boldsymbol{\tau}_{kl} \tag{25}$$

Equations 23 to 25 show that the two-point velocity correlation tensor $R_{ij}(\mathbf{r})$ can be expressed in terms of six scalar correlation functions and the Reynolds stress tensor. How well equations 23 and 24 represent a true homogeneous anisotropic turbulence field lies in the accuracy of the modeling of the scalar functions.

The true functional forms of the scalar functions for a homogeneous anisotropic velocity field is not known. A simple of the correlation functions which is based on the isotropic model will nevertheless be presented. The longitudinal correlation function can for example be modeled as

$$\widetilde{f}(r) = e^{-\pi r^2/4L^2} \tag{26}$$

where L is the longitudinal length scale in the direction of the spatial separation $\widetilde{r}_1$. Following the expressions for isotropic turbulence [4] and using equation 26 the transversal correlation functions are related to the longitudinal as

$$g(r) = h(r) = f(r) + \frac{r}{2}\frac{df(r)}{dr} = \frac{4L^2 - \pi r^2}{4L^2} e^{-\pi r^2/4L^2} \tag{27}$$

For simplicity, the cross correlation functions are modeled in the same way as the longitudinal correlation function, i.e.

$$\widetilde{f}_{12}(r) = \widetilde{f}_{13}(r) = \widetilde{f}_{23}(r) = e^{-\pi r^2/4L^2} \tag{28}$$

The functional form of the correlation functions and especially their relative shapes (equations 27 and 28) are not validated to any anisotropic turbulence. Further work is needed to develop models for these functions for a general homogeneous anisotropic turbulence.

Length scale anisotropy

Recent results from the SNGR [16] method where anisotropy has been included in the synthesis of artificial turbulence suggest that the length scale of the turbulence in different directions can be deduced from the individual components of the Reynolds stress tensor. The results indicate that the length scale in a certain direction is related to the relative magnitude of the corresponding normalized normal stress tensor component. The following relation can for example be used as a model of the length scale anisotropy for the length scale in the $\widetilde{r}_1$-direction.

$$L = L^{iso} \left(\frac{3\widetilde{\boldsymbol{\tau}}_{11}}{\widetilde{\boldsymbol{\tau}}_{11} + \widetilde{\boldsymbol{\tau}}_{22} + \widetilde{\boldsymbol{\tau}}_{33}} \right)^{1/2} \tag{29}$$

where L^{iso} is the longitudinal length scale in limit of isotropic turbulence.

Equations 23 to 29 together with the rotation matrix $\mathbf{E} = E_{ij}$ defined by the vectors in equation 19 can be used to model a general anisotropic two-point correlation tensor $R_{ij}(\mathbf{r})$ for homogeneous turbulence. The full expressions for the different components of the correlation tensor are quite large and are thus not given here.

For the isotropic case the cross correlation functions $\widetilde{f}_{12}(r)$, $\widetilde{f}_{13}(r)$ and $\widetilde{f}_{23}(r)$ as well as the off-diagonal components of the Reynolds stress tensor will be analytically zero and equations 23 and 24 reduce to the isotropic model of $R_{ij}(\mathbf{r})$ given by Batchelor [4].

$$R_{ij}(\mathbf{r}) \rightarrow R_{ij}^{iso}(\mathbf{r}) = \overline{u^2} \left(\frac{f(r) - g(r)}{r^2} r_i r_j + g(r)\delta_{ij} \right) \tag{30}$$

The accuracy of the model lies in the correctness of the expressions for the correlation functions in equations 26 to 28 and the length scale anisotropy in equation 29. One flaw of the model is that when the expressions in equations 26 to 29 are used to model the correlation functions, the resulting two-point correlation tensor $R_{ij}(\mathbf{r})$ will not fulfill the continuity condition

$$\frac{\partial R_{ij}(\mathbf{r})}{\partial r_i} = \frac{\partial R_{ij}(\mathbf{r})}{\partial r_j} = 0 \tag{31}$$

These relations can be used to constrain the models of the correlation functions in order to construct a more accurate model of the two-point velocity correlation tensor.

6 Preliminary Results

As $R_{ij}(\mathbf{r})$ is a function of the three-dimensional separation vector **r** it is difficult to illustrate. For special cases though the general features can be viewed in suitably chosen planes. Figures 5 to 9 show correlation fields from the proposed anisotropic $R_{ij}(\mathbf{r})$ in the xy-plane for $z = 0$. The interesting correlations in this slice are R_{11}, R_{22}, R_{12} where the dependence of **r** is omitted in the notation below.

The different cases are described in tables 1 and 2. They correspond to one isotropic turbulence case (case 1) and four anisotropic turbulence cases (case 2-4). The degree of anisotropy in cases 2-4 is the same but the principal axes of $\boldsymbol{\tau}_{ij}$ in cases 4 and 5 are rotated in the xy plane by 45 degrees compared to cases 2 and 3. The modeled length scale anisotropy ($L^{iso} = 0.1$) is used in cases 3 and 5 but a constant length scale ($L = 0.1$) is used in cases 2 and 4. The Reynolds stress tensors used for the different cases are given in table 2.

Case	Description
1	Isotropic
2	Anisotropic, Principal axes of $\boldsymbol{\tau}_{ij}$ 0 degrees, Isotropic length scale
3	Anisotropic, Principal axes of $\boldsymbol{\tau}_{ij}$ 0 degrees, Anisotropic length scale
4	Anisotropic, Principal axes of $\boldsymbol{\tau}_{ij}$ 45 degrees, Isotropic length scale
5	Anisotropic, Principal axes of $\boldsymbol{\tau}_{ij}$ 45 degrees, Anisotropic length scale

Table 1: Case descriptions. Isotropic length scale denotes that equation 29 has not been used.

Case	Reynolds stress tensor
1	$\boldsymbol{\tau}_{ij} = -[(400, 0, 0), (0, 400, 0), (0, 0, 400)]$
$2, 3$	$\boldsymbol{\tau}_{ij} = -[(600, 0, 0), (0, 200, 0), (0, 0, 400)]$
$4, 5$	$\boldsymbol{\tau}_{ij} = -[(400, 200, 0), (200, 400, 0), (0, 0, 400)]$

Table 2: Reynolds stress tensors $\boldsymbol{\tau}_{ij}$ expressed in x coordinate system

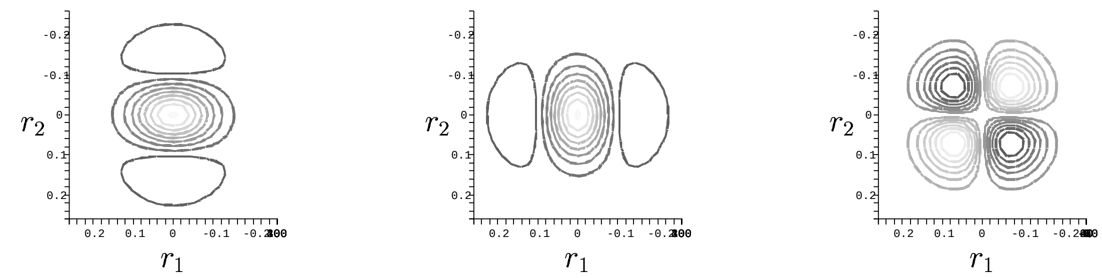
The isotropic reference case is plotted in figure 5, where R_{11} and R_{22} are identical but rotated by 90 degrees. They are positive with a Gaussian shape in the longitudinal direction but have a dip with a negative region in the transversal directions as prescribed by the isotropic model. R_{12} has two positive and two negative lobes and is zero on the coordinate axes. The correlations R_{11}, R_{22} and R_{12} are in equation 1 multiplied by the directivities in equation 7 and the isotropic correlations in case 1 result in a direction-independent sound field when the integrals are evaluated.

With an anisotropy in case 2 where $\boldsymbol{\tau}_{11} > \boldsymbol{\tau}_{22}$ the corresponding two-point correlations experience the same relations as seen in figure 6. The general shapes of

R_{11}, R_{22} and R_{12} are otherwise the same as for the isotropic case. The increased amplitude of R_{11} and decreased R_{22} will cause a sound field directivity with increased sound power in the x-direction and reduced in the y-direction.

With a length scale anisotropy (see equation 29) in case 3 the contours of R_{11}, R_{22} and R_{12} are separated in x-direction and compressed in y-direction. The length scale anisotropy emphasizes the sound power directivity mainly by making the R_{11} correlation less compact.

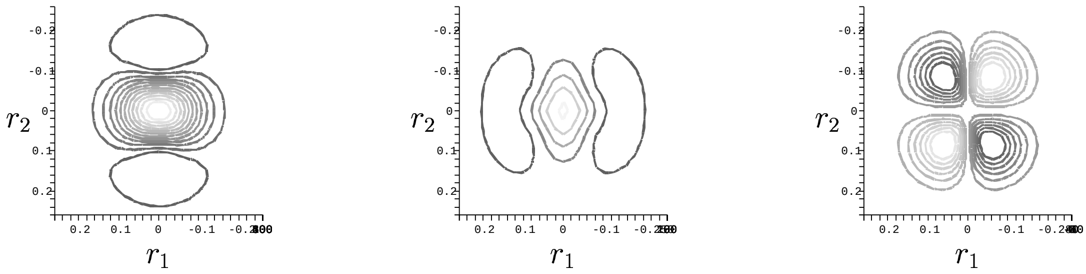
Case 4, with the same degree of anisotropy as in case 2 and 3 but with the principal axes of $\boldsymbol{\tau}_{ij}$ rotated by 45 degrees about the z-axis the R_{11}, R_{22} and R_{12} slices become as in figure 8. The R_{11} and R_{22} fields are slightly modified by the rotation of the stress tensor. The big difference is however the R_{12} field in which the positive lobe is increased and the negative lobe is decreased both in magnitude and in size. The resulting sound power directivity will through the directivities in equation 7 be increased in the 45 degree direction and decreased at -45 degrees.



(a) R_{11} (b) R_{22} (c) R_{12}

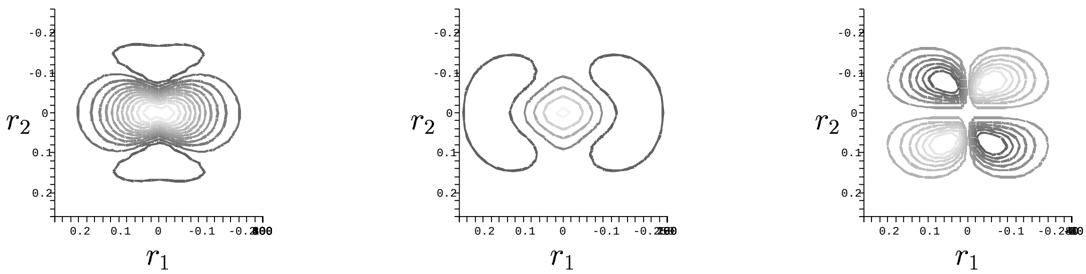
Figure 5: Case 1. Isotropic. Iso-contours. Light contours: high levels, dark contours: low (or negative) levels. Levels: a) and b) [-10:50:540]; c) [-60:10:-10 and 10:10:60]

Finally with length scale anisotropy included, the correlations are increased in the 45 degree direction and reduced at -45 degrees which again emphasizes the sound power directivity.



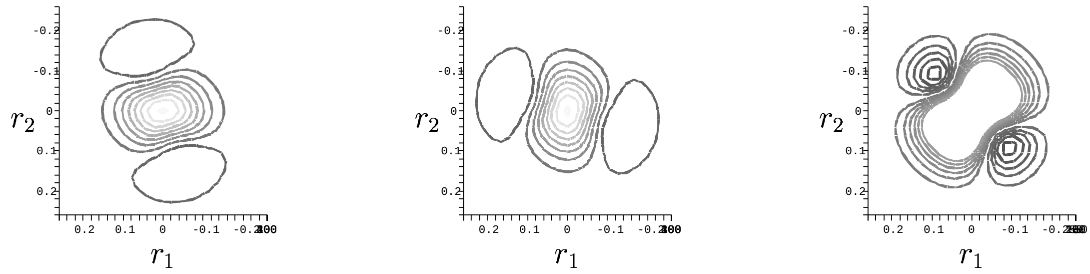
(a) R_{11} (b) R_{22} (c) R_{12}

Figure 6: Case 2. Anisotropic. Principal axes 0 degrees. Isotropic length scale. Same scaling as case 1.



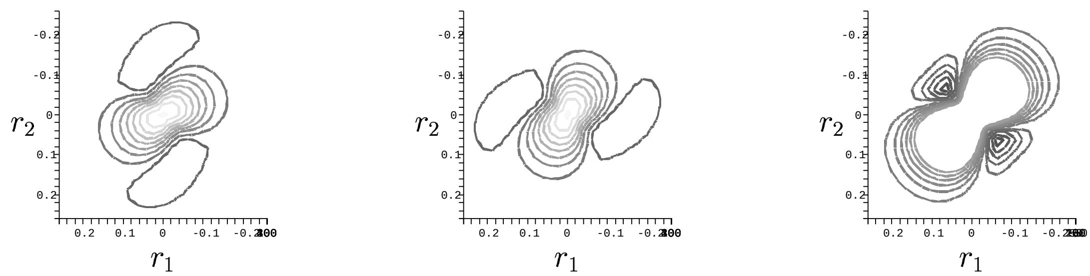
(a) R_{11} (b) R_{22} (c) R_{12}

Figure 7: Case 3. Anisotropic. Principal axes 0 degrees. Anisotropic length scale. Same scaling as case 1.



(a) R_{11} (b) R_{22} (c) R_{12}

Figure 8: Case 4. Anisotropic. Principal axes 45 degrees. Isotropic length scale. Same scaling as case 1.



(a) R_{11} (b) R_{22} (c) R_{12}

Figure 9: Case 5. Anisotropic. Principal axes 45 degrees. Anisotropic length scale. Same scaling as case 1.

7 Conclusion

In this report a first step is taken toward a general anisotropic model of the velocity correlation tensor for homogeneous turbulence. The immediate application of this model is in acoustic analogies following Lighthill where the sound directivity of the self-noise components will be properly considered through the anisotropy of the proposed model.

The anisotropic model of the two-point velocity correlation tensor includes anisotropy in terms of both velocities (Reynolds stresses) and correlations (length scales). The proposed length scale anisotropy is based on single point statistics. The length scale anisotropy is proportional to the square root of the relative magnitude of the corresponding normal Reynolds stress component.

Time scale anisotropy has not yet been included in the proposed method. A recent model of time scale anisotropy has though been presented by Jordan *et al.* [10]. The model is based on the integral and Taylor time scales which allows control of the spectral content of the turbulence.

The model is still in a developing stage and only preliminary results are presented. The results are however encouraging and the work will continue toward a closed anisotropic model of the noise generated by turbulence.

References

[1] M.J. Lighthill. On sound generated aerodynamically, i. general theory. *Proc. Roy. Soc.*, A 211:564–587, 1952.

[2] M.J. Lighthill. On sound generated aerodynamically, ii. turbulence as a source of sound. *Proc. Roy. Soc.*, A 222:1–32, 1954.

[3] H.S. Ribner. Quadrupole correlations governing the pattern of jet noise. *Journal of Fluid Mechanics*, 38:1 – 24, 1969.

[4] G.K. Batchelor. *The theory of homogeneous turbulence*. Cambridge University Press, 1953.

[5] S. Chandrasekhar. The theory of axisymmetric turbulence. *Philos. Trans. Roy. Soc.*, Vol 242. A 855:557–577, 1950.

[6] N. Andersson, L.-E. Eriksson, and L. Davidson. Large–eddy simulation of a mach 0.75 jet. Hilton Head, South Carolina, 2003. The 9th AIAA/CEAS Aeroacoustics Conference, AIAA 2003-3312.

[7] M. Goldstein and B Rosenbaum. Effect of anisotropic turbulence on aerodynamic noise. *J. Acoust. Soc. Am.*, 54 (3):630–645, 1973.

[8] A. Khavaran. Role of anisotropy in turbulent mixing noise. *AIAA Journal*, 37 , No. 7:832–841, 1999.

[9] C. Bailly, P. Lafon, and S. M. Candel. Subsonic and supersonic jet noise predictions from statistical source models. *AIAA Journal*, 35 , No. 11:1688–1696, 1997.

[10] P. Jordan, R. Wells, Y. Gervais, and J. Delville. Optimisation of correlation function models for statistical aeroacoustic noise prediction. Strasbourg, 2004. CFA/DAGA 2004 Acoustics conference.

[11] W. Bechara, P Lafon, C. Bailly, and S. M. Candel. Application of a k-epsilon turbulence model to the prediction of noise for simple and coaxial free jets. *J. Acoust. Soc. Am.*, 97(6):3518–3531, 1995.

[12] W.J. Devenport, C. Muthanna, R. Ma, S. Glegg. Two-point descriptions of wake turbulence with application to noise prediction. *AIAA Journal*, 39 , No. 12:2302–2307, 2001.

[13] P. Jordan and Y. Gervais. Modelling self and shear noise mechanisms in anisotropic turbulence. Hilton Head, South Carolina, 2003. The 9th AIAA/CEAS Aeroacooustic Conference, AIAA 2003-8743.

[14] G.M. Lilley. On the noise from jets. *AGARD CP-131*, 1974.

[15] P. Bradshaw, D H. Ferriss, and R F Johnson. Turbulence in the noise-producing region of a circular jet. *Journal of Fluid Mechanics*, vol 19:591, 1964.

[16] M. Billson, L.-E. Eriksson, and L. Davidson. Modeling of synthetic anisotropic turbulence and its sound emission. Manchester, United Kindom, 2004. The 10th AIAA/CEAS Aeroacoustics Conference, AIAA 2004-2857.