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A Note on Numerical Errors

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Contents

1 Introduction

In this note accuracy and, to some degree, stability of discretised equations is discussed.

All analysis will be conducted using either the linearised Navier-Stokes (N-S) equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} \tag{1}$$

or the hyperbolic model equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \tag{2}$$

The non-linearities of the N-S will complicate matters substantially, and hence it is chosen to consider the linearised version.

All analysis will be made using finite differences (FD). The reason is that this is considerably simpler than using the Finite Volume (FV) method. All schemes can be transfered between FD and FV on structured grids anyway, which will be shown in section 5.

For further reading, the books by Vichnevetsky [1] and Hirsch [2] can be recommended.

2 The Nature of Spatial Derivatives

In order to understand the effect of truncation errors, it is necessary to understand the effect various forms of spatial derivatives have in an equation. Consider the equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^n u}{\partial x^n} \tag{3}$$

Assume that an intermediate solution (or the initial condition) to this equation can be written as a Fourier series, where each component is

$$u(x,t) = V(t)e^{ikx} \tag{4}$$

In order to obtain a well-defined solution, this component must not grow without bounds as time progresses.

Inserting the component (4) into equation (3), and dividing by e^{ikx}, yields

$$\frac{dV(t)}{dt} = V(t)\alpha(ik)^n \tag{5}$$

Multiplication by the integrating factor $e^{-\alpha(ik)^n t}$ yields

$$\frac{dV(t)}{dt} e^{-\alpha(ik)^n t} - V(t)\alpha(ik)^n e^{-\alpha(ik)^n t} = 0 \tag{6}$$

which can be written as

$$\frac{d}{dt}\left[V(t)e^{-\alpha(ik)^n t}\right] = 0 \tag{7}$$

Integrating this yields

$$V(t) = V(0)e^{\alpha(ik)^n t} \tag{8}$$

which can be inserted into (4) to get the solution

$$u(x,t) = V(0)e^{ikx+\alpha(ik)^n t} \tag{9}$$

The effect of spatial derivatives can now be examined. First, consider the first order derivative ($n = 1$), which generates the solution

$$u(x,t) = V(0)e^{ik(x+\alpha t)} \tag{10}$$

This corresponds to translation of the solution with the phase velocity α.

The second order derivative ($n = 2$) generates the solution

$$u(x,t) = V(0)e^{-\alpha k^2 t}e^{ikx} \tag{11}$$

Hence, for $\alpha > 0$ the solution will be damped in time with no translation, since the imaginary part is constnat in time. The damping will affect short wave components (high wavenumber k) more. For $\alpha < 0$ the solution will grow without bounds.

The third order derivative ($n = 3$) generates the solution

$$u(x,t) = V(0)e^{ik(x-\alpha k^2 t)} \tag{12}$$

This is translation with the phase velocity αk^2, and hence the waves will propagate with a speed that is a function of the wavenumber.

The fourth order derivative ($n = 4$) generates the solution

$$u(x,t) = V(0)e^{\alpha k^4 t}e^{ikx} \tag{13}$$

which for $\alpha < 0$ means damping without translation, and for $\alpha > 0$ means growth without bounds.

Generally, odd order derivatives mean translation of waves, and even order derivatives mean damping or amplification, depending on the sign.

3 Truncation Errors

Approximation of a derivative with a finite difference scheme introduces errors. One way of analysing the errors is to look directly at the terms left out in the approximation.

Throughout this note, a scheme applied at node j will involve the nodes $j+l$.

3.1 Discretisation in space

Semi discretised equations will be studied, i.e. equations that are continous in time.

3.1.1 Taylor expansions and the modified equation

When either constructing a FD scheme, or analysing an existing one, one needs to write all variables u_{j+l} in terms of the central one u_j.

$$u_{j+l} = u_j + (l\Delta x)\frac{\partial u_j}{\partial x} + \frac{(l\Delta x)^2}{2!}\frac{\partial^2 u_j}{\partial x^2} + \frac{(l\Delta x)^3}{3!}\frac{\partial^3 u_j}{\partial x^3} + ... \tag{14}$$

Multiplying by a_l yields

$$a_l u_{j+l} = a_l u_j + a_l(l\Delta x)\frac{\partial u_j}{\partial x} + a_l\frac{(l\Delta x)^2}{2!}\frac{\partial^2 u_j}{\partial x^2} + a_l\frac{(l\Delta x)^3}{3!}\frac{\partial^3 u_j}{\partial x^3} + ... \tag{15}$$

Sum over all l and divide by Δx^n yields a scheme and its Taylor expansion

$$\sum_l \frac{a_l u_{j+l}}{\Delta x^n} = \sum_l \frac{a_l}{\Delta x^n}u_j + \sum_l \frac{a_l l}{\Delta x^{n-1}}\frac{\partial u_j}{\partial x} + \sum_l \frac{a_l l^2}{2!\Delta x^{n-2}}\frac{\partial^2 u_j}{\partial x^2} + \sum_l \frac{a_l l^3}{3!\Delta x^{n-3}}\frac{\partial^3 u_j}{\partial x^3} + ... \tag{16}$$

Choosing a set of a_l so that the first n terms on the right hand side cancel out will yield a finite difference scheme for the nth derivative.

The effect that the truncated terms have on the equation is seen when the so called modified equation is studied. When approximating the spatial derivatives in equation (1) with numerical schemes of type (16), the equation solved in the code is

$$\frac{\partial u_j}{\partial t} + \frac{c}{\Delta x}\sum_l a_l u_{j+l} = \frac{\nu}{\Delta x^2}\sum_l d_l u_{j+l} \tag{17}$$

Substituting the Taylor expansions of the schemes into the equation yields the modified equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} - cT_{conv} + \nu T_{visc} \tag{18}$$

where T_{conv} and T_{visc} are the truncation errors from the convective and viscous terms, respectively. It is now clear that the truncation errors affect the equation, and we will see how in the following sections.

3.1.2 The viscous term

The most obvious choice of d_l is a centred scheme. This is also the scheme that gives the highest order truncation error. A centred scheme of an even derivative means that $d_l = d_{-l}$, i.e. that the scheme is symmetric. Due to this symmetry, all odd derivatives in the Taylor expansion (16) will disappear. For example, consider the 3 point central difference approximation

$$\frac{u_{j-1} - 2u_j + u_{j+1}}{\Delta x^2} = \frac{\partial^2 u_j}{\partial x^2} + \frac{\Delta x^2}{12}\frac{\partial^4 u_j}{\partial x^4} + O(\Delta x^4) \tag{19}$$

that generates a modified equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} - cT_{conv} + \nu\frac{\Delta x^2}{12}\frac{\partial^4 u_j}{\partial x^4} + O(\Delta x^4) \tag{20}$$

This scheme is second order, but what effect does the truncation error have? If only the leading error term is considered, it is seen that this corresponds to equation (3) with $n = 4$ and $\alpha = \nu\frac{\Delta x^2}{12}$. As was shown in section 2, this represents growth of the solution without translation. The magnitude of this growth, however, is much smaller than the damping of the viscous term itself, due to the factor Δx^2.

Since the errors introduced via the approximation of the viscous term are affecting the amplitude of the solution, but much less so than the viscous term itself, only the order of the approximation is normally considered.

3.1.3 The convective term

A centred scheme of an odd derivative will yield $a_l = -a_{-l}$ and $a_0 = 0$, i.e. an anti-symmetric scheme. This anti-symmetry will cancel out all even derivatives in the Taylor expansion (16). For example, the 3 point central difference approximation

$$\frac{u_{j+1} - u_{j-1}}{2\Delta x} = \frac{\partial u_j}{\partial x} + \frac{\Delta x^2}{6}\frac{\partial^3 u_j}{\partial x^3} + O(\Delta x^4) \tag{21}$$

will generate a modified equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} + \nu T_{visc} - c\frac{\Delta x^2}{6}\frac{\partial^3 u_j}{\partial x^3} + O(\Delta x^4) \tag{22}$$

This is a second order scheme, and the truncation errors are all odd derivatives. Recall from section 2 that this corresponds to translation of waves with a phase velocity that depends on the wavenumber. This error, called dispersion error, can be dangerous, since it may mean that different waves are superpositioned due to the non-constant phase velocity. This makes the approximation of the convective term fundamentally different from that of the viscous term, since the nature of the scheme may introduce instabilities.

When dispersion errors are present, some dissipation is needed to prevent build-up of waves. There are essentially two ways of getting this dissipation, either by introducing it explicitly or by introducing it via the convective scheme.

3.1.4 Upwinded convective schemes

Upwinded schemes for the convective terms in Navier-Stokes are known to enhance stability. To see this, approximate the derivative using two schemes, one left ($a_{-1} = -1, a_0 = 1$) and one right ($a_0 = -1, a_1 = 1$) oriented. If the viscous term is left out, this yields the modified equations

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \begin{cases} +c\Delta x\frac{\partial^2 u}{\partial x^2} + O(\Delta x^2) & \text{, left oriented} \\ -c\Delta x\frac{\partial^2 u}{\partial x^2} + O(\Delta x^2) & \text{, right oriented} \end{cases} \tag{23}$$

For stability, a positive dissipative term is needed. Hence, the left oriented scheme should be chosen for $c > 0$, and the right oriented scheme for $c < 0$, and hence the name upwinding.

The fact that the scheme is non-centred gives a dissipative truncation error, and the upwinding (i.e. choosing the direction of the scheme) gives it a positive sign.

3.1.5 Jameson type artificial dissipation

An alternative way to obtain a dissipative term in the modified equation is to add it explicitly, and to use a centred scheme for the convective term. This is normally called a Jameson type scheme. Leaving out the viscous term, the equation being solved is

$$\frac{\partial u}{\partial t} + \frac{c}{\Delta x}\sum_l a_l u_{j+l} = \frac{\nu_{num}}{\Delta x^n}\sum_l d_l u_{j+l} \tag{24}$$

where n is the order of the artificial dissipation. A centred approximation of the added term yields

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = -c\ T_{conv} + \nu_{num}\frac{\partial^n u}{\partial x^n} \tag{25}$$

where the truncation error from the added term has been left out, according to section 3.1.2.

3.1.6 Jameson versus upwinding

Are the dissipative terms yielded by upwinding and Jameson type schemes similar? Choosing $\nu_{num} = \epsilon c\Delta x^{n-1}$ in the Jameson type equation (24), using a centred scheme for the convective term, yields

$$\frac{\partial u}{\partial t} + \frac{c}{\Delta x}\sum_l a_l u_{j+l} = \frac{\epsilon c}{\Delta x}\sum_l d_l u_{j+l} \tag{26}$$

which can be rewritten as

$$\frac{\partial u}{\partial t} + \frac{c}{\Delta x}\sum_l b_l u_{j+l} = 0 \tag{27}$$

where $b_l = a_l - \epsilon d_l$. Since a_l is anti-symmetric and d_l is symmetric, the resulting scheme b_l is non-centred (which in turn means that the direction of the scheme is important). Since this equation is linear, the modified equation becomes, as expected

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = -c\ T_{conv} + \epsilon c\Delta x^{n-1}\frac{\partial^n u}{\partial x^n} \tag{28}$$

Equation (28) is similar to (25), but since the dissipative term involves c, the sign of ϵ has to be chosen so that the dissipative term is indeed dissipative. Choosing the sign of ϵ is equivalent to upwinding, as was shown in section 3.1.4.

Which is to prefer, Jameson type or upwinding? Both schemes will do essentially the same, which is adding dissipation to the modified equation. There are, however, some differences.

- Computational efficiency. Upwinding involves checking the direction of propagation of information at every node prior to applying the scheme. This means *if* or *max* statements for every node in the code, which is time consuming. The Jameson type implementation, however, involves only central differences, and hence these checks are not needed. On the other hand, extra lines of code are needed to compute the added term.
- Adaptive dissipation. The dissipative term in an upwinded scheme contains a factor $c\Delta x^{n-1}$, which means that the magnitude of the term is locally dependent on the solution and the mesh. This is not the case with a Jameson scheme. For a scalar equation, like the one studied here, it is straight forward to implement an adaptive coefficient ν_{num} (e.g. $\nu_{num} = |c|\Delta x^{n-1}$). For systems of equations, however, this might not be the case. An example could be the compressible Euler equations, that support different kinds of waves that travel with different wavespeeds.

3.1.7 Choosing the amount of added dissipation

How can the correct amount of added dissipation be chosen, i.e. how can the value of ϵ be decided? One way to estimate this value is by comparing the amount of dissipation added to the viscous dissipation in N-S.

The added term in equation (28) is

$$\epsilon c\Delta x^{n-1}\frac{\partial^n u}{\partial x^n} \tag{29}$$

Recall from section 2 that this term yields solutions on the form

$$u(x,t) = V(0)e^{-\epsilon c\Delta x^{n-1}k^n t}e^{ikx} = V(0)e^{-D_a t}e^{ikx} \tag{30}$$

where

$$D_a = \epsilon c\Delta x^{n-1}k^n \tag{31}$$

Also, recall from section 2 that the viscous term in N-S yields solutions on the form

$$u(x,t) = V(0)e^{-\nu k^2 t}e^{ikx} = V(0)e^{-D_v t}e^{ikx} \tag{32}$$

where

$$D_v = \nu k^2 \tag{33}$$

Comparing the amounts of dissipation yields the normalised added dissipation

$$\bar{D}_a = \frac{D_a}{D_v} = \epsilon\frac{c\Delta x}{\nu}(k\Delta x)^{n-2} \tag{34}$$

When using a high order scheme, at least 6 computational points per wavelength are normally required to resolve a wave properly. One guideline for choosing ϵ could be that the dissipation of waves with at least 6 points per wavelength should be much smaller than the viscous dissipation of these waves. Hence, $\bar{D}_a = 0.1$ for $k\Delta x = \pi/3$ yields

$$\epsilon = 0.0832 \frac{\nu}{c\Delta x} \tag{35}$$

3.2 Discretisation in time

Discretisation of the time derivative introduces errors as well. In this section, the explicit Euler and the midpoint rule will be studied, using the model equation (2) as a base.

Using the explicit Euler scheme means that the equation solved in the code is

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{c}{\Delta x} \sum_l a_l u_{j+l}^n = 0 \tag{36}$$

where superscript n now denotes timestep. Substituting for the Taylor expansions in time gives the modified equation as

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = -\frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} + T_{conv} \tag{37}$$

Since the dissipative error term is negative, explicit Euler will diverge unless the spatial discretisation of the convective term adds enough dissipation.

When using the midpoint rule to approximate the time derivative, the equation solved is

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{c}{\Delta x} \sum_l a_l u_{j+l}^{n+\frac{1}{2}} = 0 \tag{38}$$

Substituting for the Taylor expansions in time gives the modified equation as

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = -\frac{\Delta t^2}{24}\frac{\partial^3 u}{\partial t^3} + O(\Delta t^4) + T_{conv} \tag{39}$$

The error in time is dispersive and of order Δt^2. As for the explicit Euler, the midpoint rule will need some dissipation from the spatial discretisation to converge, but less so since there is no direct amplification.

4 Fourier Analysis of Errors

A different way of studying numerical errors is by studying how a test function is affected by the discretisation, in a very similar way to what was done in section 2. Assume that the solution to the model equation (2) can be written as a Fourier series, where each component is

$$u(x,t) = V(t)e^{ikx} \tag{40}$$

Following section 2, the analytical solution to this is

$$u(x,t) = V(0)e^{ik(x-ct)} \tag{41}$$

4.1 Semi discretisation

Discretising in space only leads to the semi discretised equation at node j

$$\frac{du_j}{dt} + c\frac{1}{\Delta x}\sum_l a_l u_{j+l} = 0 \tag{42}$$

Inserting the test function (40) and dividing by e^{ikx_j} yields

$$\frac{dV(t)}{dt} + \frac{c}{\Delta x}\sum_l a_l V(t)e^{ik\Delta xl} = 0 \tag{43}$$

$V(t)$ is unaffected by the sum and can be moved out, which yields

$$\frac{d(V(t)}{dt} + V(t)\hat{A}(k) = 0 \tag{44}$$

where

$$\hat{A}(k) = \frac{c}{\Delta x}\sum_l a_l e^{ik\Delta xl} \tag{45}$$

Multiplication with the integrating factor $e^{\hat{A}(k)t}$ yields

$$\frac{d}{dt}\left[V(t)e^{\hat{A}(k)t}\right] = 0 \tag{46}$$

which can be integrated to

$$V(t) = V(0)e^{-\hat{A}(k)t} \tag{47}$$

This yields the solution

$$u_j(t) = V(0)e^{-\hat{A}(k)t}e^{ikx_j} \tag{48}$$

Splitting the real ($\Re$) and imaginary ($\Im$) parts of (48) yields

$$u_j(t) = V(0)e^{-\Re\hat{A}(k)t}e^{i(kx_j-\Im\hat{A}(k)t)} \tag{49}$$

When comparing this form of the solution to the analytical solution (41), two kinds of errors are present.

4.1.1 Amplitude errors

The amplitude of the analytical solution is constant at $V(0)$, but the amplitude of the solution to the semi discretised equation will not be constant. Instead, it will change as $e^{-\Re \hat{A}(k)t}$.

If $\Re \hat{A}(k)$ is negative, the wave will grow in time and finally diverge. If $\Re \hat{A}(k)$ is positive, the wave will dissipate in time.

The real part of $\hat{A}(k)$ can be written as

$$\Re \hat{A}(k) = \frac{c}{\Delta x} \sum_l a_l \cos(k \Delta x l) \tag{50}$$

The amplitude errors for some schemes are shown in figure 1(a). The schemes are a first order upwind, a second order central, a third order upwind, and the Dispersion Relation Preserving (DRP) scheme by Tam [3] with a 6th order derivative added for stability according to section 3.1.6. The coefficients of the schemes are listed in the appendix 6.

4.1.2 Phase velocity errors

The phase velocity c of the analytical solution (41) is constant, i.e. not dependent on the wavelength. Rewriting the solution to the semi discretised equation (49) as

$$u_j(t) = V(0) e^{-\Re \hat{A}(k)t} e^{ik(x_j - \frac{\Im \hat{A}(k)}{k} t)} \tag{51}$$

shows clearly that the phase velocity is dependent on the wavelength. Define this phase velocity as

$$c^*(k) = \frac{\Im \hat{A}(k)}{k} \tag{52}$$

and simplify to

$$c^*(k) = \frac{c}{k \Delta x} \sum_l a_l \sin(k \Delta x l) \tag{53}$$

The dispersion relation for some schemes are shown in figure 1(b). The schemes are a first order upwind, a second order central, a third order upwind, and the DRP scheme with a 6th order derivative added for stability. The coefficients of the schemes are listed in the appendix 6.

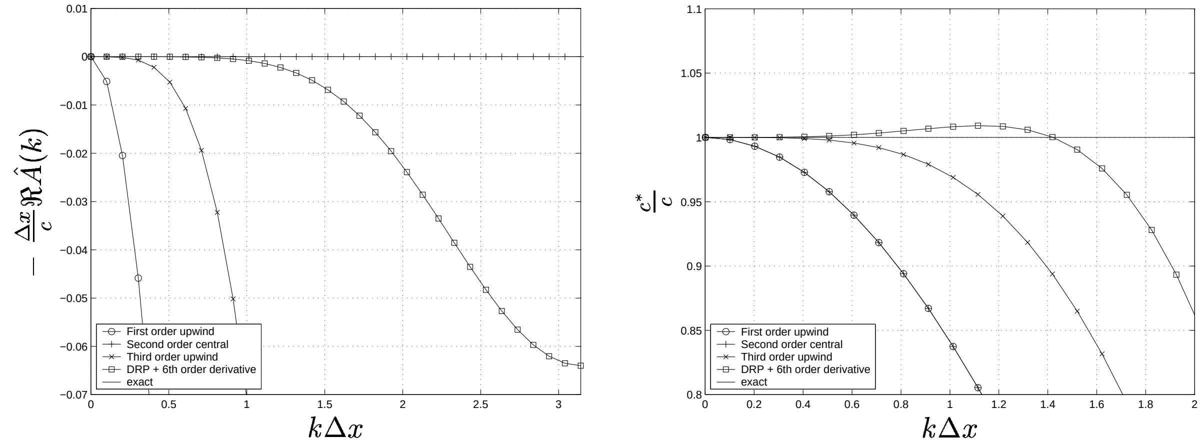
4.2 Full discretisation

When using the Crank-Nicolson scheme to discretise the time derivative, the model equation (2) becomes

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{c}{\Delta x} \frac{1}{2} \left[\sum_l a_l u_{j+l}^n + \sum_l a_l u_{j+l}^{n+1} \right] = 0 \tag{54}$$

Inserting the test function (40), simplifying, and defining the amplification function $G(k)$ yields

$$G(k) \equiv \frac{V(t^{n+1})}{V(t^n)} = \frac{1 - \frac{1}{2} \Delta t \hat{A}(k)}{1 + \frac{1}{2} \Delta t \hat{A}(k)} \tag{55}$$



(a) Dissipation relation

(b) Dispersion relation

Figure 1: Relations for semi discretisation

where $\hat{A}(k)$ is given by (45). The solution at time t^{n+1} can now be written

$$u_j(t^{n+1}) = G(k)V(t^n)e^{ikx_j} \tag{56}$$

Setting $t^n = 0$, using $G(k) = |G(k)|\, e^{i\angle G(k)}$, and rewriting the equation yields

$$u_j(\Delta t) = V(0)G(k)e^{ikx_j} = V(0)\,|G(k)|\, e^{ik(x_j - \frac{-\angle G(k)}{k\Delta t}\Delta t)} \tag{57}$$

Comparison between the expressions (57) and (41) shows directly what the errors are.

4.2.1 Amplitude errors

When scaled with the CFL number as a nondimensional timestep, the amplitude error is defined as

$$\epsilon_A = \frac{|G(k)| - 1}{\frac{c\Delta t}{\Delta x}} \tag{58}$$

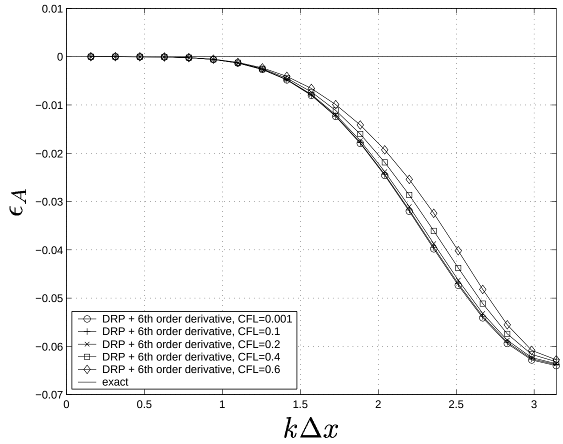
The amplitude error for the DRP scheme with a 6th order derivative added for stability together with the Crank-Nicolson scheme in time is shown in figure 2(a). The error is now dependent on the timestep, and it is seen that the difference between different timestep sizes is fairly small.

4.2.2 Phase velocity errors

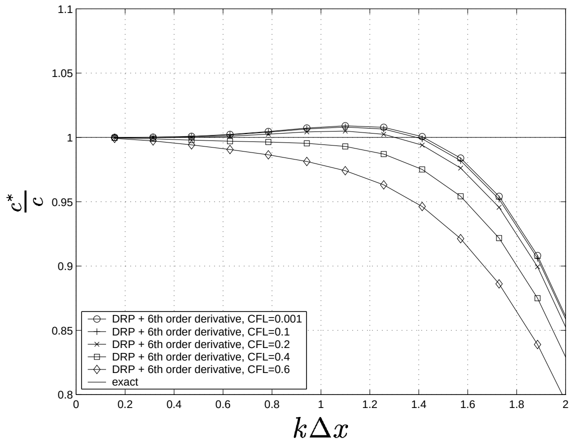
The normalised phase velocity is

$$\frac{c^*}{c} = \frac{-\angle G(k)}{kc\Delta t} = \frac{-\angle G(k)}{k\Delta x \frac{c\Delta t}{\Delta x}} \tag{59}$$

The dispersion error for the DRP scheme with a 6th order derivative added for stability together with the Crank-Nicolson scheme in time is shown in figure 2(b). The dependence on timestep size is now more apparent. For waves with 4 points per wavelength, i.e. $k\Delta x = \pi/4$, a CFL number below approximately 0.5 gives good results.



(a) Dissipation relation



(b) Dispersion relation

Figure 2: Relations for full discretisation

5 Finite Difference - Finite Volume

Finite volume (FV) schemes on structured meshes have direct equivalents in finite difference (FD) schemes. Consider the equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \tag{60}$$

which in FV, on a structured cartesian mesh, becomes

$$\frac{\partial U}{\partial t} + \frac{1}{\Delta x}\left[u_e - u_w\right] = 0 \tag{61}$$

where U is the volume average of u and u_e and u_w are the values of u at the east and west faces, respectively. Note that this is an exact relation.

Consider solving equation (60) on the meshes in figure 3 with FD and FV.

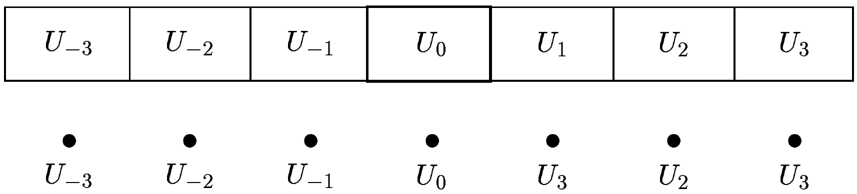


Figure 3: Corresponding FV and FD meshes

5.1 Finite differences

Using a finite difference scheme with coefficients $a_{-3} \to a_3$ to solve equation (60), the semi discretised equation becomes

$$\frac{\partial u}{\partial t} + \frac{1}{\Delta x}\sum_{l=-3}^{3} a_l u_l = 0 \tag{62}$$

5.2 Finite volume

In finite volume, the face values of u_e and u_w need to be approximated. If this is done with a 6 point scheme with coefficients $c_1 \to c_6$, the the face velocities become

$$u_e = \sum_{j=-2}^{3} c_{j+3} U_j \tag{63}$$

and

$$u_w = \sum_{j=-3}^{2} c_{j+4} U_j \tag{64}$$

Inserting these expressions into the FV equation (61) yields

$$\frac{\partial U}{\partial t} + \frac{1}{\Delta x}\left[-c_1 U_{-3} + \sum_{j=-2}^{2}(c_{j+3} - c_{j+4})U_j + c_6 U_3 + \right] = 0 \tag{65}$$

5.3 Comparison

Comparing equations (62) and (65) shows that they are similar, and the coefficients are related as

$$\begin{aligned}
a_{-3} &= -c_1 \\
a_{-2} &= c_1 - c_2 \\
a_{-1} &= c_2 - c_3 \\
a_0 &= c_3 - c_4 \\
a_1 &= c_4 - c_5 \\
a_2 &= c_5 - c_6 \\
a_3 &= c_6
\end{aligned} \tag{66}$$

6 Appendix

The coefficients for the first order upwind scheme are

$$\begin{aligned} a_{-3} &= 0 \\ a_{-2} &= 0 \\ a_{-1} &= -1 \\ a_0 &= 1 \\ a_1 &= 0 \\ a_2 &= 0 \\ a_3 &= 0 \end{aligned}$$

The coefficients for the second order central scheme are

$$\begin{aligned} a_{-3} &= 0 \\ a_{-2} &= 0 \\ a_{-1} &= -1/2 \\ a_0 &= 0 \\ a_1 &= 1/2 \\ a_2 &= 0 \\ a_3 &= 0 \end{aligned}$$

The coefficients for the third order upwind scheme are

$$\begin{aligned} a_{-3} &= 0 \\ a_{-2} &= 1/6 \\ a_{-1} &= -1 \\ a_0 &= 1/2 \\ a_1 &= 1/3 \\ a_2 &= 0 \\ a_3 &= 0 \end{aligned}$$

The coefficients for the DRP scheme by Tam [3] are

$$\begin{aligned} a_{-3} &= -0.02651995 \\ a_{-2} &= 0.18941314 \\ a_{-1} &= -0.79926643 \\ a_0 &= 0 \\ a_1 &= 0.79926643 \\ a_2 &= -0.18941314 \\ a_3 &= 0.02651995 \end{aligned}$$

The coefficients for a central approximation of a 6th order derivative are

$$
\begin{aligned}
d_{-3} &= 1 \\
d_{-2} &= -6 \\
d_{-1} &= 15 \\
d_0 &= -20 \\
d_1 &= 15 \\
d_2 &= -6 \\
d_3 &= 1
\end{aligned}
$$

References

[1] Vichnevetsky R., Bowles J. B. Fourier analysis of numerical approximations of hyperbolic equations. 1982.

[2] Hirsch C. Numerical computation of internal and external flows. 1, 1988.

[3] Tam C. K. W., Webb J. C. Dispersion-relation-preserving finite difference schemes for computational acoustics. *J. Comp. Physics*, 107:262–281, 1993.