TDMA Solver

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In this chapter we'll derive the TDMA (Tri-Diagonal Matrix Algorithm). The solver is really a formula for recursive use of solving a matrix equation using Gauss-Elimination. The finite volume discretization gives a tri-diagonal (the diagonal plus two off-diagonals) equation system in 1D, a penta-diagonal system in 2D, and a septa-diagonal system in 3D. Some discretization schemes give more diagonals; for example, QUICK gives seven in 2D. In this case one can simply put the two outermost diagonals in the source term, so that a penta-diagonal equation system is retained.

Below the TDMA for 2D is given in detail, and the extension to 3D is straight forward.

The 2D discretized equation reads

$$a_P T_P = a_E T_E + a_W T_W + a_N T_N + a_S T_S + S_U. \tag{24}$$

We rewrite it as

$$a_i T_i = b_i T_{i+1} + c_i T_{i-1} + d_i \tag{25}$$

and identification gives

$$\begin{aligned} &a_i = a_P,\ b_i = a_E,\ c_i = a_W \\ &d_i = a_N T_N + a_S T_S + S_U. \end{aligned}$$

Equation 25 is solved from $i = 2$ to $i = ni - 1$, and $i = 1$ and $i = ni$ are boundary nodes. We want to write Eq. 25 on the form

$$T_i = P_i T_{i+1} + Q_i. \tag{26}$$

In order to derive Eq. 26 we write Eq. 25 on matrix form so that

$$\begin{bmatrix} a_2 & -b_2 & 0 & \ldots & & \\ -c_3 & a_3 & -b_3 & 0 & \ldots & \\ 0 & -c_4 & a_4 & -b_4 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} T_2 \\ T_3 \\ T_4 \\ \vdots \end{bmatrix} = \begin{bmatrix} d_2 + c_2 T_1 \\ d_3 \\ d_4 \\ \vdots \end{bmatrix} \tag{27}$$

Start by dividing the first row by a_2 so that (see Eq. 26)

$$\begin{bmatrix} 1 & -P_2 & 0 & \ldots & & \\ -c_3 & a_3 & -b_3 & 0 & \ldots & \\ 0 & -c_4 & a_4 & -b_4 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} T_2 \\ T_3 \\ T_4 \\ \vdots \end{bmatrix} = \begin{bmatrix} Q_2 \\ d_3 \\ d_4 \\ \vdots \end{bmatrix} \tag{28}$$

where

$$P_2 = \frac{b_2}{a_2},\; Q_2 = \frac{d_2 + c_2 T_1}{a_2}. \tag{29}$$

Now we want to eliminate the $c's$. Multiply row 1 by c_3, add it to row 2 and after that divide row 2 by $a_3 - c_3 P_2$. We obtain

$$\begin{bmatrix} 1 & -P_2 & 0 & \ldots & & \\ 0 & 1 & -P_3 & 0 & \ldots & \\ 0 & -c_4 & a_4 & -b_4 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} T_2 \\ T_3 \\ T_4 \\ \vdots \end{bmatrix} = \begin{bmatrix} Q_2 \\ Q_3 \\ d_4 \\ \vdots \end{bmatrix} \tag{30}$$

where

$$P_3 = \frac{b_3}{a_3 - c_3 P_2},\quad Q_3 = \frac{d_3 + c_3 Q_2}{a_3 - c_3 P_2}. \tag{31}$$

We see that Eq. 31 becomes an recursive equation for P_i and Q_i on the form

$$P_i = \frac{b_i}{a_i - c_i P_{i-1}}, \quad Q_i = \frac{d_i + c_i Q_{i-1}}{a_i - c_i P_{i-1}}. \tag{32}$$

Now the P_i and Q_i coefficients can be computed.

1. for $i = 2$: use Eq. 29.
2. for $i = 3$ to $i = ni - 1$: use Eq. 32.

Compute T from Eq. 26 starting from $i = ni - 1$ to $i = 2$.