2.1 – 2.5. Equations

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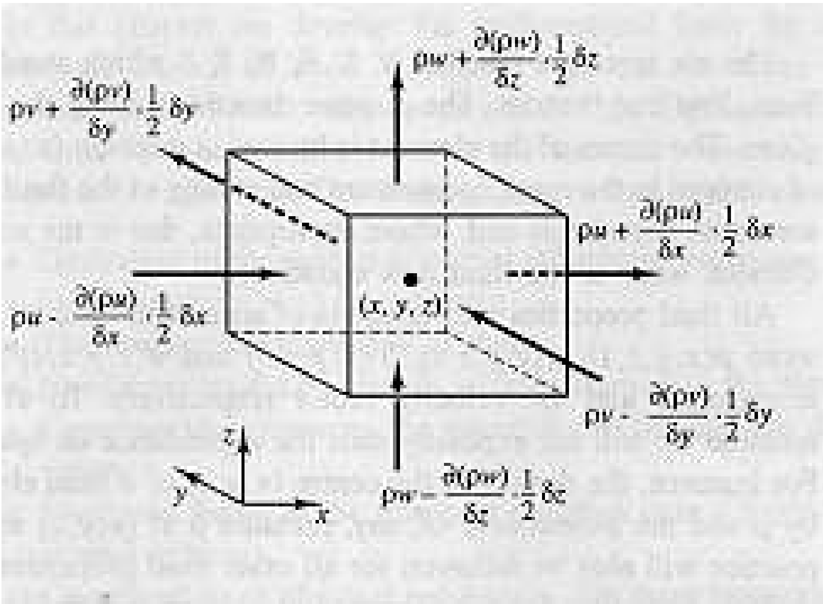
The governing equations are derived. Note the balance principle:

out - in = internal generation

You can regard all equations like this, even though the Navier-Stokes equations usually are interpreted as Newton's second law, which states that mass times acceleration is equal to the sum of external forces.



The *general* transport equation reads

$$\frac{\partial(\rho\phi)}{\partial t} + div(\rho\mathbf{u}\phi) = div(\Gamma grad\phi) + S_\phi$$

In finite volume methods this equations is integrated, and the volume integrals are transformed to area integrals using Gauss' law so that (steady form)

$$\int_A \rho \mathbf{u} \cdot \mathbf{n} dA = \int_A \Gamma grad\phi \cdot \mathbf{n} dA + \int_V S_\phi dA$$

This equation states: the *net* transport (i.e. in minus out) due to convection is equal to the neat transport due to diffusion, plus source terms.

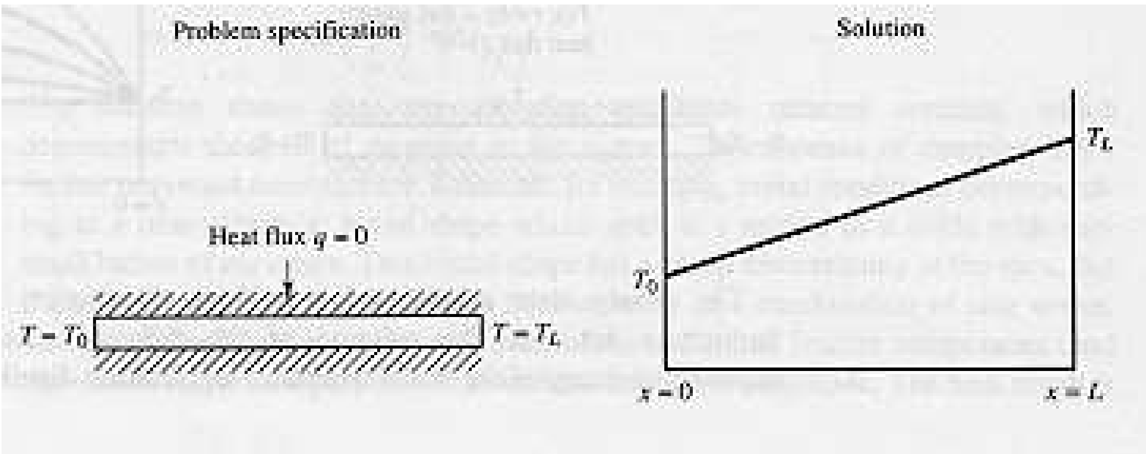
2.6. Classification

The equations are classified according to how disturbances of the variable propagate.

- Steady problems

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

This problem is *elliptic*, which means that disturbances in one point propagate throughout the whole domain. One simple example is steady, 1D, heat conduction (without any heat generation), see the figure below (fig. 2.6)



for which the governing equation reads

$$k\frac{d^2T}{dx^2} = 0$$

Boundary conditions are needed at all (two) boundaries. A sudden change in a boundary conditions spreads throughout the whole rod. For Dirichlet boundary conditions at $x = 0$ and $x = L$ the solution is a straight line (see figure above).

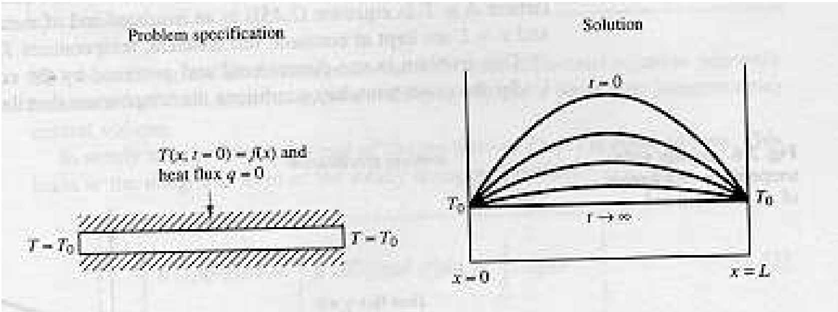
- Marching problems

These can either be *parabolic* or *hyperbolic*.

An example of a parabolic equation is the time-dependent heat conduction

$$\frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial x^2}$$

with boundary conditions $T = 0$ at $x = 0$ and $x = L$. Initially the rod is uniformly heated, and at $t = 0$ the heat generation is switched off, see the figure below (fig. 2.7)



The steady state consists of a uniform distribution of temperature $T = T_0$ throughout the rod. The solution of the diffusion equation (2.46) yields the exponential decay of the initial quadratic temperature distribution. Initial conditions

As can be seen from Fig. 2.7 the temperature decreases for increasing time. That the problem is parabolic means that we have one-way influence with respect to one (or more) independent coordinates. In this example time t is the parabolic coordinate. The past influence the future, but not vice versa.

Often a space coordinate is parabolic. In boundary-layer types of flow, the streamwise coordinate x is a one-way coordinate. The influence goes from upstream to downstream, but not (or a least very weakly) in the other direction. This is because the streamwise convection (parabolic behavior) is much stronger than the streamwise diffusion (elliptic behavior)

A good example of an hyperbolic equation is the wave equation

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \frac{\partial^2 \phi}{\partial x^2}$$

where c denotes speed of sound.

2.7 The Role of Characteristics

The wave equation can through the transformation $\xi = x - ct$ and $\eta = x + ct$ be written

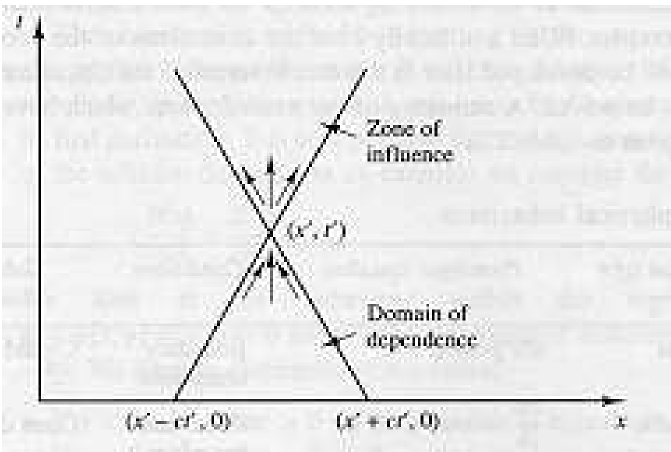
$$\frac{\partial^2 \phi}{\partial \xi \partial \eta} = 0$$

The solution of this equation is only dependent of the initial values of ϕ. At point x_1, t_1, for example, $\phi(x_1, t_1)$ depends only on the initial values between $x_1 - ct_1$ and $x_1 + ct_1$, see the figure below (fig. 2.9) The solution is given by

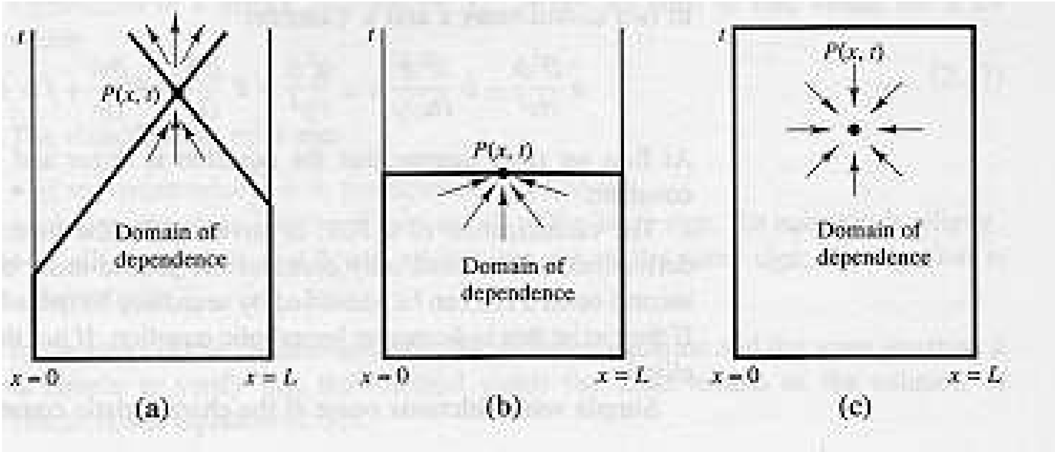
$$\phi(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct) +\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)dx$$

where f and g are initial conditions at $t = 0$ given by

$$\phi(x,0) = f(x)$$
$$\frac{\partial \phi(x,0)}{\partial t} = g(x)$$



The propagation of disturbances to a point P for hyperbolic, parabolic and elliptic equations are summarized in the figure below (fig. 2.10)



• Some examples of boundary conditions for incompressible flow are shown in the figure below (fig. 2.13)