Low-Re Number Models

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In low-Re number models (LRN models) many grid lines should be located in the near-wall region (see Chapter 4 in LD). Usually the first node should be located at $y^+ \simeq 1$, and $5-10$ nodes up to $y^+ \simeq 20$. N.B.: the term "low Reynolds number" refers to the local, turbulent Reynolds number

$$Re_\ell = \frac{\mathcal{U}\ell}{\nu} \propto \frac{\nu_t}{\nu}.$$

That Re_ℓ is small means that viscous effects are important, which occurs close to walls. It has nothing to do with the global Re numbers Re_D, Re_L, Re_x, etc.

In LRN models we want to make sure that the modelled terms in the k and ε equations behave in the same way as their exact counterparts when $y \to 0$. Taylor expansion of the fluctuating velocities (also valid for the time averaged $\bar{U}_i$ and the instantaneous velocities U_i)

$$\begin{aligned} u &= a_0 + a_1 y + a_2 y^2 \ldots \\ v &= b_0 + b_1 y + b_2 y^2 \ldots \\ w &= c_0 + c_1 y + c_2 y^2 \ldots \end{aligned} \tag{84}$$

At the wall (i.e. at $y = 0$) vi have no-slip, i.e. $u = v = w = 0$, which gives $a_0 = b_0 = c_0 = 0$. Furthermore, at the wall

$$\frac{\partial u}{\partial x} = \frac{\partial w}{\partial z} = 0. \tag{85}$$

The continuity equation for the fluctuating velocities (incompressible flow) reads

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

which, together with Eq. 85 gives $\partial v/\partial y = 0$. Equation 84 now gives $b_1 = 0$. Thus we have the following behavior of the velocities near a wall

$$\begin{array}{lll} u & = a_1 y + & a_2 y^2 \ldots \\ v & = & b_2 y^2 \ldots \\ w & = c_1 y + & c_2 y^2 \ldots \end{array} \tag{86}$$

Now we proceed to compare modelled and exact terms when $y \to 0$.

The production term

$$\begin{array}{lll} \text{exact} & -\overline{uv}\dfrac{\partial \bar{U}}{\partial y} & = \mathcal{O}\left(y^3\right) \times \mathcal{O}\left(y^0\right) = \mathcal{O}\left(y^3\right) \\ \text{modelled} & \nu_t \left(\dfrac{\partial \bar{U}}{\partial y}\right)^2 & = \mathcal{O}\left(y^4\right) \times \mathcal{O}\left(y^0\right) = \mathcal{O}\left(y^4\right) \end{array} \tag{87}$$

The first line in the above equation is obtained directly by insertion of Eq. 86. For the second line, we first need to establish how $\nu_t \propto k^2/\varepsilon$ varies near the wall. The dissipation is defined as

$$\varepsilon = \nu \overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}}.$$

As $y \to 0$ we have that $\partial/\partial y \gg \partial/\partial x, \partial/\partial z$ so that the dissipation can be written as

$$\varepsilon \simeq \nu \overline{\frac{\partial u_i}{\partial y}\frac{\partial u_i}{\partial y}} = \nu \left(\overline{\frac{\partial u}{\partial y}\frac{\partial u}{\partial y}} + \overline{\frac{\partial v}{\partial y}\frac{\partial v}{\partial y}} + \overline{\frac{\partial w}{\partial y}\frac{\partial w}{\partial y}} \right)$$

From Eq. 86 we find that the two largest terms are the derivatives of u and w, i.e.

$$\varepsilon \simeq \nu \left(\overline{a_1^2 + c_1^2 + \ldots} \right) = \mathcal{O}\left(y^0\right). \tag{88}$$

Furthermore, Eq. 86 gives

$$k = \frac{1}{2}\left(\overline{u^2} + \overline{v^2} + \overline{w^2}\right) = \frac{1}{2}\left(\overline{a_1^2 y^2} + \overline{b_2^2 y^4} + \overline{c_1^2 y^2 + \ldots}\right) = \mathcal{O}\left(y^2\right), \tag{89}$$

and thus

$$\nu_t = c_\mu \frac{k^2}{\varepsilon} = \mathcal{O}\left(y^4\right) \tag{90}$$

which gives line two in Eq. 87.

Note that when we compared the behavior of the exact and modelled production terms, in reality we compared the shear stresses, i.e.

$$\begin{aligned} &\text{exact} && -\overline{uv} && = \mathcal{O}\left(y^3\right) \\ &\text{modelled} && \nu_t \frac{\partial \bar{U}}{\partial y} && = \mathcal{O}\left(y^4\right) \end{aligned} \tag{91}$$

We want to modify the modelled shear stress so that it behaves like $\mathcal{O}(y^3)$. We can do that by introducing a function f_μ. In this case it should be of the form $f_\mu = \mathcal{O}(y^{-1})$. (Note that this is an unusual form of damping function since $f_\mu \to \infty$ when $y \to 0$; ususally the damping functions $\to 0$ when $y \to 0$). The turbulent viscosity should now be computed as

$$\nu_t^{LR} = c_\mu f_\mu \frac{k^2}{\varepsilon},$$

where upper index LR denotes Low Reynolds number. Note that the damping term should be devised so that $f_\mu \to 1$ in the log region, i.e. for $y^+ > 30$.

One example of a damping function is $f = 1 - \exp(-R_T^{0.25})$. Let's see how f behaves when $y \to 0$. Recall that Taylor expansion for $\exp(-x)$ reads

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \ldots$$

Thus we get for f

$$f = 1 - e^{-R_T^{0.25}} = 1 - \left(1 - R_T^{0.25} + \frac{\left(R_T^{0.25}\right)^2}{2!} - \ldots\right) = R_T^{0.25} - \frac{R_T^{0.5}}{2} - \ldots$$

Since $R_T \propto \nu_t/\nu$, we get from Eq. 90

$$f = \mathcal{O}(y^1)$$

The diffusion term

The exact diffusion term includes two parts: triple correlations and pressure diffusion. From experiments and DNS (Direct Numerical Simulations) we know that the first part is much larger than the second. Thus:

$$\begin{array}{lll} \text{exact} & \overline{vk'} & \mathcal{O}\left(y^4\right) \\ \text{modelled} & \dfrac{\nu_t^{LR}}{\sigma_k}\dfrac{\partial k}{\partial y} & \mathcal{O}\left(y^3\right) \times \mathcal{O}\left(y^1\right) = \mathcal{O}\left(y^4\right) \end{array} \tag{92}$$

where we have used $\nu_t^{LR} = \mathcal{O}(y^3)$. As can be seen, the exact and the modelled term both behave as $\mathcal{O}(y^4)$.

Wall B.C. for ε

Above we found that at the wall $\varepsilon = \mathcal{O}(y^0)$. This presents a problem: how should we specify ε_{wall}?

- k equation

One way is to use the k equation. As $y \to 0$, only two terms remain in the k equation, namely the viscous diffusion term and the dissipation term so that ($\sigma_k = 1$)

$$0 = \nu \frac{\partial^2 k}{\partial y^2} - \varepsilon,$$

which gives

$$\varepsilon_{wall} = \nu \left(\frac{\partial^2 k}{\partial y^2} \right)_{wall}.$$

However, this type of boundary condition can be numerically unstable, since it relies on the evaluation of a second derivative at the wall.

- Taylor expansion

From Eqs. 88 and 89 we have

$$\begin{aligned} \varepsilon &= \nu \left(\overline{a_1^2 + c_1^2} \right) \\ k &= \frac{1}{2} \left(\overline{a_1^2 y^2} + \overline{c_1^2 y^2} \right). \end{aligned} \tag{93}$$

Take the derivative of $\sqrt{k}$ with respect to y which gives

$$\begin{aligned} \sqrt{k} &= \frac{1}{\sqrt{2}} \left(\overline{a_1 y} + \overline{c_1 y} \right) \\ \Rightarrow \left(\frac{\partial \sqrt{k}}{\partial y} \right)^2 &= \frac{1}{2} \left(\overline{a_1^2} + \overline{c_1^2} \right). \end{aligned}$$

From Eq. 93 we now find that

$$\varepsilon_{wall} = 2\nu \left(\frac{\partial \sqrt{k}}{\partial y} \right)^2_{wall}$$

Solving for $\tilde{\varepsilon}$

Another option is to add a term D in the k equation (see Eq. 79)

$$\frac{\partial \bar{U}k}{\partial x} + \frac{\partial \bar{V}k}{\partial y} = \frac{\partial}{\partial y}\left[\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial y}\right] + \nu_t\left(\frac{\partial \bar{U}}{\partial y}\right)^2 - \underbrace{(\tilde{\varepsilon} + D)}_{\varepsilon},$$

where D is chosen so that $D_{wall} = \varepsilon_{wall}$, and thus $\tilde{\varepsilon}_{wall} = 0$. In the Launder & Sharma model (see Section 4 in LD)

$$D = 2\nu\left(\frac{\partial\sqrt{k}}{\partial y}\right)^2.$$

The turbulent viscosity is computed as

$$\nu_t = c_\mu \frac{k^2}{\tilde{\varepsilon}},$$

and since $\tilde{\varepsilon} = \mathcal{O}(y)$, we get $\nu_t = \mathcal{O}(y^3)$. As a consequence no damping term f_μ is needed, since in this way

$$-\overline{uv} = \nu_t \frac{\partial \bar{U}}{\partial y} = \mathcal{O}\left(y^3\right),$$

which is the same as the exact term, see Eq. 91.

However, note that some turbulence models in which $\tilde{\varepsilon}$ is solved still use a $f_\mu = \mathcal{O}y^0$ in order to improve the results.