Mechanics of solids and fluids

-Introduction to continuum mechanics

by

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Introduction to continuum mechanics

1. Tensors

1.1 Index notation

Before introducing concepts of tensor algebra we introduce the index notation. The index notation simplifies writing of quantities as well as equations and will be used in the remaining of this text. There are two types of indices:

- *Free indices* are only used once per quantity and can take the integer values 1, 2 and 3. For example for one free index i:

$$
\begin{aligned}
a_i \quad &\Leftrightarrow \quad a_1,\ a_2 \text{ and } a_3 \\
a_i = b_i \quad &\Leftrightarrow \quad a_1 = b_1,\ a_2 = b_2 \text{ and } a_3 = b_3
\end{aligned}
$$

  Similarly we can have two (or more) free indices i and j:

$$
\begin{aligned}
a_{ij} \quad &\Leftrightarrow \quad a_{11},\ a_{12},\ a_{13},\ a_{21},\ a_{22},\ a_{23},\ a_{31},\ a_{32} \text{ and } a_{33} \\
a_{ij} = b_{ij} \quad &\Leftrightarrow \quad a_{11} = b_{11},\ a_{12} = b_{12},\ a_{13} = b_{13},\ \ldots,\ a_{32} = b_{32} \text{ and } a_{33} = b_{33}
\end{aligned}
$$

- *Summation indices* are used twice per term and indicates a summation of that index from 1 to 3. For example:

$$
\begin{aligned}
a_{ii} \quad &\Leftrightarrow \quad \sum_{i=1}^{3} a_{ii} \\
a_i\, b_i \quad &\Leftrightarrow \quad \sum_{i=1}^{3} a_i\, b_i \\
a_{ij}\, b_{ij} \quad &\Leftrightarrow \quad \sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij} b_{ij}
\end{aligned}
$$

  This sum over repeated indices is often called Einstein's summation convention.

Often these two types of indices are used together. A simple example is the equation system

$$
a_i = T_{ij}\, b_j \;\Leftrightarrow\; a_1 = \sum_{j=1}^{3} T_{1j}\, b_j,\ a_2 = \sum_{j=1}^{3} T_{2j}\, b_j \text{ and } a_3 = \sum_{j=1}^{3} T_{3j}\, b_j
$$

where i is a free index and j a summation index. Another way to express this equation system is to use *matrices* (in this example two column matrices 3×1 and a square matrix

3×3):

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{1.1}$$

which by using index notation can be written as:

$$[a_i] = [T_{ij}]\,[b_j] \tag{1.2}$$

Problem 1 Explain the following symbols: A_{ii}, A_{ijj}, A_{ij}, $a_i A_{ij}$, $c_i b_j A_{ij}$.
For each index tell whether it is a summation/dummy index or a free index.

Problem 2 Use index notation to re-write the following expression: $f_1 u_1 + f_2 u_2 + f_3 u_3$
Answer: $f_i\, u_i$

Problem 3 Expand $a_{ijk}\, b_{ik}$ by giving the terms explicitly.
Answer: $a_{i1k}\, b_{ik} = \sum_{i=1}^{3} \sum_{k=1}^{3} a_{i1k}\, b_{ik} = a_{111}\, b_{11} + a_{112}\, b_{12} + a_{113}\, b_{13} + a_{211}\, b_{21} + a_{212}\, b_{22} + a_{213}\, b_{23} + a_{311}\, b_{31} + a_{312}\, b_{32} + a_{313}\, b_{33}$,
$a_{i2k}\, b_{ik} = \sum_{i=1}^{3} \sum_{k=1}^{3} a_{i2k}\, b_{ik} = a_{121}\, b_{11} + a_{122}\, b_{12} + a_{123}\, b_{13} + a_{221}\, b_{21} + a_{222}\, b_{22} + a_{223}\, b_{23} + a_{321}\, b_{31} + a_{322}\, b_{32} + a_{323}\, b_{33}$,
$a_{i3k}\, b_{ik} = \sum_{i=1}^{3} \sum_{k=1}^{3} a_{i3k}\, b_{ik} = a_{131}\, b_{11} + a_{132}\, b_{12} + a_{133}\, b_{13} + a_{231}\, b_{21} + a_{232}\, b_{22} + a_{233}\, b_{23} + a_{331}\, b_{31} + a_{332}\, b_{32} + a_{333}\, b_{33}$

Matlab example 1 An example of using Matlab commands for matrix definitions (for T and b) and multiplication $a_i = T_{ij}\, b_j$ is given below:

```
>> T=[1 2 3; 4 5 6; 7 8 9];
>> b=[1 2 3]';
>> a=T*b
a =
    14
    32
    50
```

Python example 1 An example of using Python commands for matrix definitions (for T and b) and multiplication $a_i = T_{ij}\, b_j$ is given below:

```
import numpy as np
T = np.array([[1.,2.,3.],[4.,5.,6.],[7.,8.,9.]])
b=np.array([1.,2.,3.])
a=T@b
```

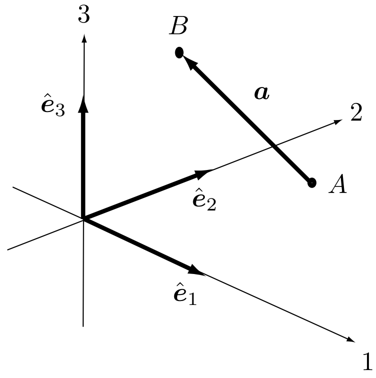


Figure 1.1: Illustration of vector $\boldsymbol{a}$.

```
print(a)
[14. 32. 50.]
```

1.2 Vectors

Orthonormal base vectors

To describe many physical quantities (such as force, displacement, velocity) both magnitude and direction must be given. Hence, these quantities can be described by vectors (1st order tensors) in a 3-dimensional Euclidean space. By introducing a set of right-handed orthonormal basis vectors $\{\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3\}$ any vector $\boldsymbol{a} = \overrightarrow{AB}$ can be expressed as a linear combination these basis vectors, $\hat{\boldsymbol{e}}_i$:

$$\boldsymbol{a} = a_1\hat{\boldsymbol{e}}_1 + a_2\hat{\boldsymbol{e}}_2 + a_3\hat{\boldsymbol{e}}_3 = a_i\hat{\boldsymbol{e}}_i. \tag{1.3}$$

as shown in Figure 1.1. The coefficients a_i or (a_1, a_2, a_3) are the *components* of $\boldsymbol{a}$ with respect to the basis $\hat{\boldsymbol{e}}_i$. The length (=Euclidean norm) of a vector $\boldsymbol{a}$ is denoted a or $|\boldsymbol{a}|$. For normalized vectors (describing only direction) the following notations are introduced:

$$\hat{\boldsymbol{e}}_a = \hat{\boldsymbol{a}} = \frac{\boldsymbol{a}}{a}, \tag{1.4}$$

whereby a vector $\boldsymbol{a}$ can be written as $\boldsymbol{a} = a\,\hat{\boldsymbol{e}}_a$. Examples of normalized vectors are the basis vectors $\{\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3\}$.

■ **Example 1.1** Assume that the vector $\boldsymbol{a} = a_1\hat{\boldsymbol{e}}_1 + a_2\hat{\boldsymbol{e}}_2 + a_3\hat{\boldsymbol{e}}_3 = a_i\hat{\boldsymbol{e}}_i$. The normalized vector $\hat{\boldsymbol{a}}$ is obtained as follows:

$$\hat{\boldsymbol{a}} = \frac{a_1\hat{\boldsymbol{e}}_1 + a_2\hat{\boldsymbol{e}}_2 + a_3\hat{\boldsymbol{e}}_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}}$$

■

Problem 4 Determine the unit length vector along $\boldsymbol{a} = 4\,\hat{\boldsymbol{e}}_1 + 6\,\hat{\boldsymbol{e}}_2 - 12\,\hat{\boldsymbol{e}}_3$.
Answer: $\hat{\boldsymbol{a}} = 2/7\,\hat{\boldsymbol{e}}_1 + 3/7\,\hat{\boldsymbol{e}}_2 - 6/7\,\hat{\boldsymbol{e}}_3$

Scalar product

To each pair of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ there corresponds a real number $\boldsymbol{a} \cdot \boldsymbol{b}$, called the scalar product. The scalar product is defined as (see Figure 1.2):

$$\boldsymbol{a} \cdot \boldsymbol{b} = a\,b\,\cos\theta \tag{1.5}$$

where θ is the angle between the vectors.

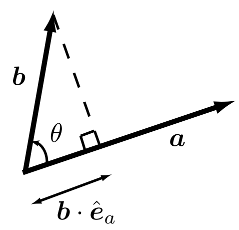


Figure 1.2: Illustration of scalar product.

The vector projection of $\boldsymbol{b}$ on $\boldsymbol{a}$ is defined as the orthogonal projection of $\boldsymbol{b}$ on a line parallel to $\boldsymbol{a}$ and is equal to $b\,\cos(\theta)$. It can be obtained from the definition of scalar product as $\boldsymbol{b} \cdot \hat{\boldsymbol{e}}_a$. Further, it is possible to express the length of a vector $a = |\boldsymbol{a}|$ as follows

$$a = |\boldsymbol{a}| = \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}} \tag{1.6}$$

By now applying the scalar product between the orthonormal basis vectors $\{\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3\}$, the following results are obtained

$$\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j = \delta_{ij} \tag{1.7}$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{when} \quad i = j \\ 0 & \text{when} \quad i \neq j \end{cases} \tag{1.8}$$

The symbol δ_{ij} is called the *Kronecker delta symbol*. The scalar product between vectors is a bilinear operator and has the following properties:

$$\begin{cases} \boldsymbol{a} \cdot (\alpha \boldsymbol{b} + \beta \boldsymbol{c}) & = \alpha\,\boldsymbol{a} \cdot \boldsymbol{b} + \beta\,\boldsymbol{a} \cdot \boldsymbol{c} \\ \\ (\alpha\,\boldsymbol{a} + \beta\,\boldsymbol{b}) \cdot \boldsymbol{c} & = \alpha\,\boldsymbol{a} \cdot \boldsymbol{c} + \beta\,\boldsymbol{b} \cdot \boldsymbol{c} \end{cases}$$

where α and β are scalars. These properties can now be used to show that the scalar product between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ may be written as:

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_i \hat{\boldsymbol{e}}_i \cdot b_j \hat{\boldsymbol{e}}_j = a_i b_j \hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j = a_i b_j \delta_{ij} = a_i b_i \tag{1.9}$$

and that the scalar product is commutative i.e. $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$. The components a_i of a vector $\boldsymbol{a}$ can be extracted by scalar multiplication with corresponding base vectors $\hat{\boldsymbol{e}}_i$ which is shown by the following derivation:

$$\hat{\boldsymbol{e}}_i \cdot \boldsymbol{a} = \hat{\boldsymbol{e}}_i \cdot a_j \hat{\boldsymbol{e}}_j = a_j \, \hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j = a_j \, \delta_{ij} = a_i \tag{1.10}$$

Problem 5 Compute the projection of the vector $\boldsymbol{a} = 4\,\hat{\boldsymbol{e}}_1 + 6\,\hat{\boldsymbol{e}}_2 - 12\,\hat{\boldsymbol{e}}_3$ on the line defined by the vector $\boldsymbol{b} = 1\,\hat{\boldsymbol{e}}_1 + 1\,\hat{\boldsymbol{e}}_2 + 1\,\hat{\boldsymbol{e}}_3$
Answer: $-2/\sqrt{3}$

Problem 6 Expand the following expressions of the Kronecker delta δ_{ij}:
$\delta_{ij}\delta_{ij}$, $\delta_{ij}\delta_{jk}\delta_{ki}$, $\delta_{ij}A_{ik}$
Answers: 3, 3, A_{jk}.

Vector product

Another product that is useful is the vector product $\boldsymbol{a}\times\boldsymbol{b}$, which is illustrated in Figure 1.3. The result is a vector that is orthogonal to the plane spanned by $\boldsymbol{a}$ and $\boldsymbol{b}$ (with a right-handed system) and has the length

$$|\boldsymbol{a} \times \boldsymbol{b}| = a\, b \sin\theta \tag{1.11}$$

By applying the vector product to the orthonormal basis vectors $\{\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3\}$, the following results are obtained

$$\hat{\boldsymbol{e}}_i \times \hat{\boldsymbol{e}}_j = e_{ijk} \hat{\boldsymbol{e}}_k, \tag{1.12}$$

where the *permutation symbol* e_{ijk} is defined as

$$e_{ijk} = \begin{cases} 1 & \text{when} \quad ijk = 123,\ 231 \text{ or } 312 \\ -1 & \text{when} \quad ijk = 321,\ 213 \text{ or } 132 \\ 0 & \text{otherwise} \end{cases} \tag{1.13}$$

The vector product is bilinear, i.e.

$$\begin{cases} \boldsymbol{a} \times (\alpha \boldsymbol{b} + \beta \boldsymbol{c}) & = \alpha\, \boldsymbol{a} \times \boldsymbol{b} + \beta\, \boldsymbol{a} \times \boldsymbol{c} \\ (\alpha\, \boldsymbol{a} + \beta\, \boldsymbol{b}) \times \boldsymbol{c} & = \alpha\, \boldsymbol{a} \times \boldsymbol{c} + \beta\, \boldsymbol{b} \times \boldsymbol{c} \end{cases}$$

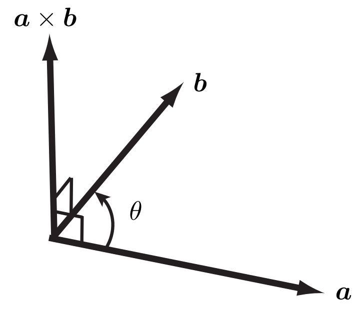


Figure 1.3: Illustration of vector product.

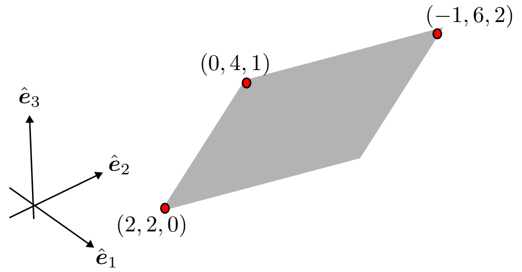
whereby the vector product between two arbitrary vectors becomes

$$\boldsymbol{a} \times \boldsymbol{b} = (a_i \hat{\boldsymbol{e}}_i) \times (b_j \hat{\boldsymbol{e}}_j) = a_i b_j \hat{\boldsymbol{e}}_i \times \hat{\boldsymbol{e}}_j = a_i b_j e_{ijk} \hat{\boldsymbol{e}}_k. \tag{1.14}$$

From this we can note the relation $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$ which is found by using the properties of the permutation symbol. The permutation symbol and the Kronecker's delta symbol are linked by the so-called e-δ identity:

$$e_{ijm} e_{klm} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}. \tag{1.15}$$

Problem 7 Compute the unit normal to the plane



Answer: $\hat{\boldsymbol{n}} \approx \pm(-0.45\,\hat{\boldsymbol{e}}_2 + 0.89\,\hat{\boldsymbol{e}}_3)$

Problem 8 Show that $e_{ijk}\delta_{jk} = 0$

Problem 9 Prove that for three arbitrary vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ the following relation holds:

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\,\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\,\boldsymbol{c}$$

Open product

Open product (also called outer product) between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ results in a 2nd order tensor $\boldsymbol{T}$ (also called dyad) as follows

$$\boldsymbol{a}\,\boldsymbol{b} = a_i\hat{\boldsymbol{e}}_i\, b_j\hat{\boldsymbol{e}}_j = a_i\, b_j\, \hat{\boldsymbol{e}}_i\, \hat{\boldsymbol{e}}_j = T_{ij}\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_j = \boldsymbol{T} \tag{1.16}$$

The open product is bilinear but not cummutative i.e. $\boldsymbol{a}\,\boldsymbol{b} \neq \boldsymbol{b}\,\boldsymbol{a}$ in general. 2nd order tensors will be further exploited in Section 1.3. In literature the open product is sometimes for clarity denoted by $\otimes$, i.e., the dyad is written as $\boldsymbol{a} \otimes \boldsymbol{b}$.

Column matrix representation of components

In a given coordinate system defined by the basis vectors $\{\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3\}$, the vector components a_i can be collected in a column matrix as follows

$$[\boldsymbol{a}] = [a_i] = [\, a_1\ a_2\ a_3\,]^{\mathrm{T}} \tag{1.17}$$

An example is the base vector $\hat{\boldsymbol{e}}_1$ that is represented by the following column matrix

$$[\hat{\boldsymbol{e}}_1] = [\,1\ 0\ 0\,]^{\mathrm{T}} \tag{1.18}$$

Therefore, the scalar multiplication between two vectors can be obtained as

$$\boldsymbol{a}\cdot\boldsymbol{b} = a_i b_i = [\boldsymbol{a}]^{\mathrm{T}}\,[\boldsymbol{b}]\,. \tag{1.19}$$

■ **Example 1.2** Assume that $\boldsymbol{a} = a_i\hat{\boldsymbol{e}}_i$ and $\boldsymbol{b} = b_i\hat{\boldsymbol{e}}_i$. The matrix representation of the components of the dyad $\boldsymbol{a}\,\boldsymbol{b}$ is given as:

$$[\boldsymbol{a}\,\boldsymbol{b}] = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{bmatrix}$$

■

Matlab example 2 Example of scalar product in Matlab

```
>> a=[1 2 3]'; b=[3 4 5]';
>> c=sum(a.*b)
c =
    26
```

Example of cross product in Matlab

```
>> a=[1 2 3]'; b=[3 4 5]';
>> c=cross(a,b)
```

```
c =
    -2
     4
    -2
```

Python example 2 Example of scalar and cross product in Python

```
import numpy as np
a=np.array([1.,2.,3.])
b=np.array([3.,4.,5.])
print(a@b)
print(np.cross(a,b))
26.0
[-2.  4. -2.]
```

Coordinate system transformation

A vector must be invariant with respect to coordinate system. Assume two different sets of orthonormal basis vectors $\{\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3\}$ and $\left\{\hat{\boldsymbol{e}}'_1, \hat{\boldsymbol{e}}'_2, \hat{\boldsymbol{e}}'_3\right\}$. The vector $\boldsymbol{b}$ can then be written as

$$\boldsymbol{b} = b_i \hat{\boldsymbol{e}}_i = b'_i \hat{\boldsymbol{e}}'_i \tag{1.20}$$

The components b'_i can be extracted from $\boldsymbol{b}$ as

$$b'_i = \hat{\boldsymbol{e}}'_i \cdot \boldsymbol{b} = \hat{\boldsymbol{e}}'_i \cdot b_j \, \hat{\boldsymbol{e}}_j = \hat{\boldsymbol{e}}'_i \cdot \hat{\boldsymbol{e}}_j \, b_j \tag{1.21}$$

In matrix notation this can be written

$$[b'_i] = [l_{ij}] \, [b_j] = \left[\hat{\boldsymbol{e}}'_i \cdot \hat{\boldsymbol{e}}_j\right] [b_j] \tag{1.22}$$

where the transformation matrix $[l_{ij}]$ is orthogonal, i.e. $[l_{ij}]^{\mathrm{T}} = [l_{ij}]^{-1}$. This can be understood if we assume that the components b'_j are known and then the components b_i can be extracted from $\boldsymbol{b}$ (similarly to (1.21)) as

$$b_i = \hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}'_j \, b'_j \tag{1.23}$$

Matlab example 3 Example of Matlab input file to define e_{ijk}-operator and vector product $c_k = a_i\, b_j\, e_{ijk}$:

```
%definition of permutation symbol
perm=zeros(3,3,3);
for i=1:3
    for j=1:3
        for k=1:3
%%%
if ( (i==1) & (j==2) & (k==3)) | ((i==2) & (j==3) & (k==1)) | ...
     ((i==3) & (j==1) & (k==2))
                perm(i,j,k)=1;
elseif ( (i==3) & (j==2) & (k==1)) | ((i==2) & (j==1) & (k==3)) | ...
         ((i==1) & (j==3) & (k==2))
                perm(i,j,k)=-1;
end
%%%
        end
    end
end
%computation of vector product c_k= a_i b_j perm_ijk
a=[1 2 3]';
b=[4 5 6]';
c=zeros(3,1);
for k=1:3
    c(k)=0;
    for i=1:3
        for j=1:3
            c(k)=c(k,1)+a(i)*b(j)*perm(i,j,k);
        end
    end
end
```

Python example 3 Example of Python file to define e_{ijk}-operator and compute vector product $c_k = a_i\, b_j\, e_{ijk}$:

```
import numpy as np
```

```
def permutation(i,j,k):
    f=0
    i=i+1; j=j+1; k=k+1; #due to first index=0 in Python
    if (i==1 and j==2 and k==3):
        f=1
    elif (i==2 and j==3 and k==1):
        f=1
    elif (i==3 and j==1 and k==2):
        f=1
    elif (i==1 and j==3 and k==2):
        f=-1
    elif (i==2 and j==1 and k==3):
        f=-1
    elif (i==3 and j==2 and k==1):
        f=-1
    return f

a=np.array([1.,2.,3.])
b=np.array([3.,4.,5.])
c=np.zeros((3))

for k in range(3):
    for i in range(3):
        for j in range(3):
            c[k]=c[k]+a[i]*b[j]*permutation(i,j,k)

print(c)
[-2.  4. -2.]
```

1.3 2nd order tensors

Representation of 2nd order tensors

2nd order tensors are physical quantitites that describe how vectors change with e.g. direction and position in space. Examples of 2nd order tensors that we will explore later are the stress tensor, strain tensor, velocity gradient and the deformation gradient. A 2nd order tensor $\boldsymbol{T}$ is represented in a orthonormal coordinate system $\{\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3\}$ as

$$\boldsymbol{T} = T_{ij}\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_j, \tag{1.24}$$

where T_{ij} are the nine *components* of $\boldsymbol{T}$ and $\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_j$ are the *base dyads*. The base dyads $\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_j$ are 2nd order tensors themselves and $\boldsymbol{T}$ is built up by a linear combination of these scaled by the components T_{ij}.

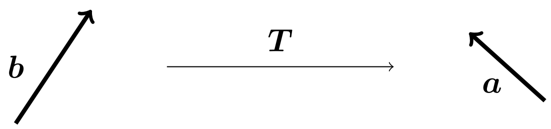
■ **Example 1.3** The matrix representation of the components of $\boldsymbol{T}$ are obtained as

$$[\boldsymbol{T}] = T_{ij}\,[\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_j] = T_{11}\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} + T_{12}\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} + \ldots = \begin{bmatrix}T_{11} & T_{12} & T_{13}\\ T_{21} & T_{22} & T_{23}\\ T_{31} & T_{32} & T_{33}\end{bmatrix}$$

■

Scalar product between 2nd order tensors and vectors

Now consider a linear transformation of a vector $\boldsymbol{b}$ into $\boldsymbol{a}$



This may be written symbolically by introducing a scalar product as

$$\boldsymbol{a} = \boldsymbol{T}\cdot\boldsymbol{b} \tag{1.25}$$

where the linear operator $\boldsymbol{T}$ is a second-order tensor. Before proceeding we define the scalar product between a base dyad $\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_j$ and a base vector $\hat{\boldsymbol{e}}_k$ as:

$$(\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_j)\cdot\hat{\boldsymbol{e}}_k = \hat{\boldsymbol{e}}_i\,(\hat{\boldsymbol{e}}_j\cdot\hat{\boldsymbol{e}}_k) = \delta_{jk}\hat{\boldsymbol{e}}_i \quad \text{and } \hat{\boldsymbol{e}}_i\cdot(\hat{\boldsymbol{e}}_j\hat{\boldsymbol{e}}_k) = (\hat{\boldsymbol{e}}_i\cdot\hat{\boldsymbol{e}}_j)\,\hat{\boldsymbol{e}}_k = \delta_{ij}\,\hat{\boldsymbol{e}}_k \tag{1.26}$$

■ **Example 1.4** Two example of results are: $(\hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_2)\cdot\hat{\boldsymbol{e}}_2 = \hat{\boldsymbol{e}}_1$ and $\hat{\boldsymbol{e}}_1\cdot(\hat{\boldsymbol{e}}_2\hat{\boldsymbol{e}}_2) = \boldsymbol{0}$ ■

The scalar product between a 2nd order tensor and a vector is assumed to be bilinear. If we use index notations then such a scalar product can be written as:

$$\boldsymbol{a} = \boldsymbol{T}\cdot\boldsymbol{b} = T_{ij}\,\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_j\cdot b_k\,\hat{\boldsymbol{e}}_k = T_{ij}b_k\hat{\boldsymbol{e}}_i\,\delta_{jk} = \underbrace{T_{ij}b_j}_{a_i}\,\hat{\boldsymbol{e}}_i = a_i\,\hat{\boldsymbol{e}}_i \tag{1.27}$$

Often we omit the basis and simply write the relation between the components, i.e.

$$a_i = T_{ij}\,b_j \tag{1.28}$$

It is then implicitly assumed that the same basis vectors are used for all variables. Furthermore, standard matrix manipulations can be used in numerical implementations to compute the components a_i

$$\begin{bmatrix}a_1\\ a_2\\ a_3\end{bmatrix} = \begin{bmatrix}T_{11} & T_{12} & T_{13}\\ T_{21} & T_{22} & T_{23}\\ T_{31} & T_{32} & T_{33}\end{bmatrix}\begin{bmatrix}b_1\\ b_2\\ b_3\end{bmatrix}$$

If we switch the order of the vector and the 2nd order tensor in the scalar product

$$\boldsymbol{c} = \boldsymbol{b} \cdot \boldsymbol{T} = b_i\, \hat{\boldsymbol{e}}_i \cdot T_{jk}\, \hat{\boldsymbol{e}}_j \hat{\boldsymbol{e}}_k = b_i\, T_{jk}\, \delta_{ij}\, \hat{\boldsymbol{e}}_k = \underbrace{b_j T_{jk}}_{c_k} \hat{\boldsymbol{e}}_k = c_k\, \hat{\boldsymbol{e}}_k \tag{1.29}$$

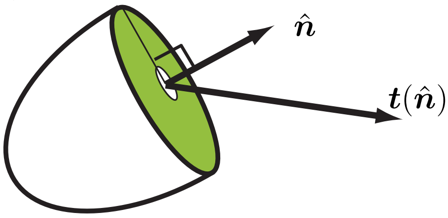
or in short $c_k = b_j\, T_{jk}$. A consequence of these results is that:

$$\boldsymbol{T} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{T}^{\mathrm{T}}$$

where the transpose of the tensor is defined as $\boldsymbol{T}^{\mathrm{T}} = T_{ji}\, \hat{\boldsymbol{e}}_i \hat{\boldsymbol{e}}_j$. In matrix notations and operations this corresponds to

$$\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1 \; b_2 \; b_3 \end{bmatrix} \begin{bmatrix} T_{11} & T_{21} & T_{31} \\ T_{12} & T_{22} & T_{32} \\ T_{13} & T_{23} & T_{33} \end{bmatrix}$$

An example of a 2nd order tensor is the stress tensor $\boldsymbol{\sigma}$ from which the traction vector $\boldsymbol{t}(\hat{\boldsymbol{n}})$ can be obtained as $\boldsymbol{t}(\hat{\boldsymbol{n}}) = \hat{\boldsymbol{n}} \cdot \boldsymbol{\sigma}$.



■ **Example 1.5** By using this scalar multiplication twice it is possible to find the components of a 2nd order tensor T_{ij} by

$$\hat{\boldsymbol{e}}_i \cdot \boldsymbol{T} \cdot \hat{\boldsymbol{e}}_j = \hat{\boldsymbol{e}}_i \cdot (T_{kl} \hat{\boldsymbol{e}}_k \hat{\boldsymbol{e}}_l) \cdot \hat{\boldsymbol{e}}_j = T_{kl} \delta_{ik} \delta_{lj} = T_{ij} \tag{1.30}$$

■

A special 2nd order tensor is the identity tensor $\boldsymbol{\delta}$:

$$\boldsymbol{\delta} = \delta_{ij} \hat{\boldsymbol{e}}_i \hat{\boldsymbol{e}}_j = \hat{\boldsymbol{e}}_i \hat{\boldsymbol{e}}_i \tag{1.31}$$

with the property that it does not transform a vector $\boldsymbol{a}$ when scalar multiplied with $\boldsymbol{\delta}$, i.e. $\boldsymbol{\delta} \cdot \boldsymbol{a} = \boldsymbol{a}$ and $\boldsymbol{a} \cdot \boldsymbol{\delta} = \boldsymbol{a}$ (or written by using only the components $\delta_{ij}\, a_j = a_j\, \delta_{ji} = a_i$). The components for δ can be collected in the following matrix form:

$$[\boldsymbol{\delta}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.32}$$

Multiplication between 2nd order tensors

Scalar multiplication (also called single contraction) between two base dyads is defined as

$$(\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_j)\cdot(\hat{\boldsymbol{e}}_k\hat{\boldsymbol{e}}_l) = \hat{\boldsymbol{e}}_i\,(\hat{\boldsymbol{e}}_j\cdot\hat{\boldsymbol{e}}_k)\,\hat{\boldsymbol{e}}_l = \hat{\boldsymbol{e}}_i\,\delta_{jk}\,\hat{\boldsymbol{e}}_l = \delta_{jk}\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_l \tag{1.33}$$

This scalar multiplication is assumed to be bilinear and therefore the scalar multiplication of two 2nd order $\boldsymbol{T}$ and $\boldsymbol{U}$ can be written as

$$\boldsymbol{T}\cdot\boldsymbol{U} = T_{ij}\,\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_j\cdot U_{kl}\hat{\boldsymbol{e}}_k\hat{\boldsymbol{e}}_l = T_{ij}\,U_{kl}\,\hat{\boldsymbol{e}}_i\,\delta_{jk}\,\hat{\boldsymbol{e}}_l = \underbrace{T_{ij}\,U_{jl}}_{V_{il}}\,\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_l = \boldsymbol{V} \tag{1.34}$$

or in terms of components $T_{ij}\,U_{jk} = V_{ik}$. Hence, when using a matrix notation then the components of $\boldsymbol{V}$ can simply be obtained by a standard matrix multiplication between $[T_{ij}]$ and $[U_{jk}]$. Further, by applying the transpose operator to such a product it can be shown that

$$\boldsymbol{V}^{\mathrm{T}} = (\boldsymbol{T}\cdot\boldsymbol{U})^{\mathrm{T}} = \boldsymbol{U}^{\mathrm{T}}\cdot\boldsymbol{T}^{\mathrm{T}} \tag{1.35}$$

■ **Example 1.6** To show (1.35) we use the index notation:

$$V_{ij}^{\mathrm{T}} = V_{ji} = T_{jk}\,U_{ki} = U_{ik}^{\mathrm{T}}\,T_{kj}^{\mathrm{T}}$$

■

Another operator that we introduce is the double contraction operator between two base dyads

$$(\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_j):(\hat{\boldsymbol{e}}_k\hat{\boldsymbol{e}}_l) = (\hat{\boldsymbol{e}}_i\cdot\hat{\boldsymbol{e}}_k)\,(\hat{\boldsymbol{e}}_j\cdot\hat{\boldsymbol{e}}_l) = \delta_{ik}\,\delta_{jl} \tag{1.36}$$

If we assume bilinearity of that operator then double contraction between two 2nd order $\boldsymbol{T}$ and $\boldsymbol{U}$ results in a scalar α and is obtained as

$$\boldsymbol{T}:\boldsymbol{U} = T_{ij}\,U_{ij} = \alpha \tag{1.37}$$

Problem 10 If $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors and $\boldsymbol{A}$ and $\boldsymbol{B}$ are 2nd order tensors show that
a) $(\boldsymbol{a}\cdot\boldsymbol{A})\cdot\boldsymbol{b} = \boldsymbol{a}\cdot(\boldsymbol{A}\cdot\boldsymbol{b})$
b) $(\boldsymbol{A}\cdot\boldsymbol{B})^{\mathrm{T}} = \boldsymbol{B}^{\mathrm{T}}\cdot\boldsymbol{A}^{\mathrm{T}}$
c) $(\boldsymbol{A}\cdot\boldsymbol{a})\cdot(\boldsymbol{B}\cdot\boldsymbol{b}) = \boldsymbol{a}\cdot(\boldsymbol{A}^{\mathrm{T}}\cdot\boldsymbol{B})\cdot\boldsymbol{b}$

Problem 11 The components of the 2nd order tensors and vectors are given as:

$$[A_{ij}] = \begin{bmatrix}1 & 2 & 0\\ 2 & 3 & 4\\ 0 & 4 & 2\end{bmatrix},\quad [B_{ij}] = \begin{bmatrix}3 & 0 & 0\\ 0 & 3 & 1\\ 0 & 1 & 2\end{bmatrix},\quad [a_i] = \begin{bmatrix}2\\ 3\\ 1\end{bmatrix},\quad [b_i] = \begin{bmatrix}1\\ -1\\ 2\end{bmatrix}$$

Compute

a) $\boldsymbol{A} \cdot \boldsymbol{a}$
b) $\boldsymbol{a} \cdot \boldsymbol{b}$
c) $\boldsymbol{A} : \boldsymbol{B}$
d) $\boldsymbol{A} : (\boldsymbol{a}\,\boldsymbol{b})$
Answers: a) $[8, 17, 14]^{\mathrm{T}}$, b) 1, c) 24, d) 19.

Matlab example 4 An example of computing the double contraction in Matlab:

```
>> T=[1 2 3; 4 5 6; 7 8 9];
>> U=[1 2 1; 3 4 3; 5 6 5];
>> alpha=sum(sum(T.*U))
alpha =
   186
```

Python example 4 An example of computing the double contraction in Python:

```
import numpy as np
T = np.array([[1.,2.,3.],[4.,5.,6.],[7.,8.,9.]])
U = np.array([[1.,2.,1.],[3.,4.,3.],[5.,6.,5.]])
S=T*U
alpha=S.sum()
print(alpha)
186.0
```

Symmetric and skew-symmetric 2nd order tensors

As introduced earlier the transpose $\boldsymbol{T}^{\mathrm{T}}$ of a 2nd order tensor $\boldsymbol{T}$ is defined as follows:

$$\boldsymbol{T}^{\mathrm{T}} = T_{ij}\hat{\boldsymbol{e}}_j\hat{\boldsymbol{e}}_i = T_{ji}\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_j \tag{1.38}$$

Many second-order tensors in mechanics are symmetric which means that the tensor and its transpose are equal e.g. $\boldsymbol{T}^{\mathrm{T}} = \boldsymbol{T}$ or in components $T_{ij} = T_{ji}$. Another type of tensors is the skew-symmetric second-order tensors. These have the property that the transpose of the tensor is equal to the tensor with a minus sign, e.g., $\boldsymbol{T}^{\mathrm{T}} = -\boldsymbol{T}$ or $T_{ij} = -T_{ji}$. Clearly, for such a tensor the diagonal elements (in a matrix representation) must be equal to zero whereby the components can be collected in the following general matrix

$$\begin{bmatrix} 0 & T_{12} & T_{13} \\ -T_{12} & 0 & T_{23} \\ -T_{13} & -T_{23} & 0 \end{bmatrix} \tag{1.39}$$

Problem 12 Show that $A_{ij} = e_{ijk}a_k$ is skew-symmetric (i.e. $A_{ji} = -A_{ij}$).

Problem 13 If A_{ij} is symmetric and B_{ij} is skew-symmetric. Show that $A_{ij}B_{ij} = 0$.

Inverse of a 2nd order tensor

If we assume that the tensor $\boldsymbol{T}$ gives the linear transformation $\boldsymbol{a} = \boldsymbol{T} \cdot \boldsymbol{b}$ between the two vectors $\boldsymbol{b}$ and $\boldsymbol{a}$. Then we can introduce the inverse $\boldsymbol{T}^{-1}$ of this transformation as $\boldsymbol{b} = \boldsymbol{T}^{-1} \cdot \boldsymbol{a}$. If we express these two relations in components

$$a_i = T_{ij}\, b_j \quad \text{and } b_i = T^{-1}_{ij}\, a_j \tag{1.40}$$

then it is obvious that the components of the $\boldsymbol{T}^{-1}$ can be found using standard matrix inversion i.e.

$$\left[T^{-1}_{ij}\right] = \left[T_{ij}\right]^{-1} \tag{1.41}$$

Hence, standard rules for matrix inversion apply also for tensor components such as $(\boldsymbol{T} \cdot \boldsymbol{U})^{-1} = \boldsymbol{U}^{-1} \cdot \boldsymbol{T}^{-1}$.

Coordinate system transformation

A 2nd order tensor is invariant with respect to coordinate system. Assume two different sets of orthonormal basis vectors $\{\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3\}$ and $\left\{\hat{\boldsymbol{e}}'_1, \hat{\boldsymbol{e}}'_2, \hat{\boldsymbol{e}}'_3\right\}$. The 2nd order tensor $\boldsymbol{T}$ can then be written with either basis vectors:

$$\boldsymbol{T} = T_{ij}\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_j = T'_{ij}\hat{\boldsymbol{e}}'_i\hat{\boldsymbol{e}}'_j \tag{1.42}$$

The components T'_{ij} can be extracted from $\boldsymbol{T}$ as

$$T'_{ij} = \hat{\boldsymbol{e}}'_i \cdot \boldsymbol{T} \cdot \hat{\boldsymbol{e}}'_j = \hat{\boldsymbol{e}}'_i \cdot \hat{\boldsymbol{e}}_k\, T_{kl}\, \hat{\boldsymbol{e}}_l \cdot \hat{\boldsymbol{e}}'_j \tag{1.43}$$

In matrix notation this can be written

$$\left[T'_{ij}\right] = \left[l_{ik}\right]\left[T_{kl}\right]\left[l_{jl}\right]^{\mathrm{T}} \tag{1.44}$$

where the orthogonal transformation matrix $[l_{ij}]$ was defined in (1.22).

Higher order tensors

It is possible to construct tensors of any order (or rank) as follows:

$$\boldsymbol{A} = A_{ijk\cdots}\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_j\hat{\boldsymbol{e}}_k\cdots$$

In particular, fourth-order tensors are frequently used to, for example, give the relation (material behavior) between the second-order tensors stress and strain.

Problem 14 a) Determine the transformation matrix when $\hat{\boldsymbol{e}}'_1$ is parallel to $\hat{\boldsymbol{e}}_1+\hat{\boldsymbol{e}}_2-\hat{\boldsymbol{e}}_3$ and $\hat{\boldsymbol{e}}'_2$ parallel to $\hat{\boldsymbol{e}}_2+\hat{\boldsymbol{e}}_3$.

Answer:

$$[l_{ij}] = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}$$

b) The components of the 2nd order stress tensor $\boldsymbol{\sigma}$ have been measured by engineer Emil in the coordinate system $\{\hat{\boldsymbol{e}}_i\}$ with the following results

$$[\sigma_{ij}] = \begin{bmatrix} 400 & 100 & 0 \\ 100 & 200 & 0 \\ 0 & 0 & 300 \end{bmatrix} \quad \text{MPa}$$

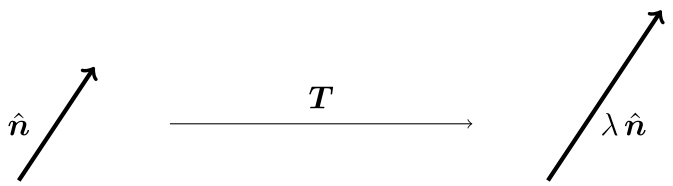
Engineer Emilia uses the coordinate system $\{\hat{\boldsymbol{e}}'_i\}$ (from a)), what are the stress components in that coordinate system?

Answer:

$$\left[\sigma'_{ij}\right] = \begin{bmatrix} 367 & 0 & 94.3 \\ 0 & 250 & 86.6 \\ 94.3 & 86.6 & 283 \end{bmatrix} \quad \text{MPa}$$

1.4 Principal values and principal directions

A second-order tensor can be, as discussed above, thought of a linear transformation between vectors, i.e. $\boldsymbol{a} = \boldsymbol{T}\cdot\boldsymbol{b}$. Important properties of a second-order tensor are its eigenvectors (principal directions) and eigenvalues (principal values). Eigenvectors are defined as vectors that do not rotate upon transformation with the second-order tensor. If $\hat{\boldsymbol{n}}$ now is an eigenvector to $\boldsymbol{T}$ this can be illustrated as



This can be written as

$$\lambda\,\hat{\boldsymbol{n}} = \boldsymbol{T}\cdot\hat{\boldsymbol{n}} \quad \text{or } \lambda\hat{n}_i = T_{ij}\,\hat{n}_j. \tag{1.45}$$

The eigenvectors $\hat{\boldsymbol{n}}$ are chosen to be of unit length whereby it is possible to identify the length of the vector $\boldsymbol{T}\cdot\hat{\boldsymbol{n}}$ as the corresponding eigenvalues λ. The way to find the eigenvalues and eigenvectors is to rewrite (1.45) as

$$(\lambda\,\boldsymbol{\delta} - \boldsymbol{T}) \cdot \hat{\boldsymbol{n}} = \mathbf{0} \quad \text{or} \quad (\lambda\,\delta_{ij} - T_{ij})\,\hat{n}_j = 0_i. \tag{1.46}$$

A trivial solution to this equation is that $\hat{\boldsymbol{n}} = \mathbf{0}$. However, it is possible to find non-trivial solution if $(\lambda\,\boldsymbol{\delta} - \boldsymbol{T})$ is non-invertible. From linear algebra we know that then the determinant of the matrix $[\lambda\,\boldsymbol{\delta} - \boldsymbol{T}]$ must be zero, i.e.

$$\det\left(\boldsymbol{T} - \lambda\,\boldsymbol{\delta}\right) = 0 \tag{1.47}$$

which is called the characteristic equation. An important theorem from linear algebra is the spectral theorem which states that for symmetric matrices the eigenvalues are real and the eigenvectors are orthogonal. In the current course we will only consider eigenvalues and eigenvectors for symmetric second order tensors (i.e. stress, strain, etc) and for such a tensor the characteristic equation can be obtained as

$$\begin{vmatrix} T_{11}-\lambda & T_{12} & T_{13} \\ T_{12} & T_{22}-\lambda & T_{23} \\ T_{13} & T_{23} & T_{33}-\lambda \end{vmatrix} = (T_{11}-\lambda)\,(T_{22}-\lambda)\,(T_{33}-\lambda) +$$
$$T_{12}\,T_{23}\,T_{13} + T_{13}\,T_{12}\,T_{23} - T_{13}^2(T_{22}-\lambda) - T_{23}^2\,(T_{11}-\lambda) - (T_{33}-\lambda)\,T_{12}^2 = 0$$

This third order polynomial equation can be summarized as

$$\lambda^3 - I_1\,\lambda^2 + I_2\,\lambda - I_3 = 0 \tag{1.48}$$

where the invariants of the second-order tensor $\boldsymbol{T}$ were introduced as

$$I_1 = T_{ii}, \quad I_2 = \left[T_{ii}\,T_{jj} - T_{ij}T_{ij}\right]/2, \quad I_3 = \det(T_{ij}) \tag{1.49}$$

After solving the three eigenvalues $\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}$ from (1.48) we can solve the corresponding eigenvectors $\hat{\boldsymbol{n}}^{(1)}, \hat{\boldsymbol{n}}^{(2)}, \hat{\boldsymbol{n}}^{(3)}$ from (1.46).

Problem 15 Find eigenvalues and eigenvectors of:

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Answers:

$$\lambda^{(1)} \approx 2.41 \quad \lambda^{(2)} = 1 \quad \lambda^{(3)} \approx -0.414$$

$$\hat{\boldsymbol{n}}^{(1)} = \pm\,[-0.924\ 0.383\ 0]^{\mathrm{T}} \quad \hat{\boldsymbol{n}}^{(2)} = \pm\,[0\ 0\ 1]^{\mathrm{T}} \quad \hat{\boldsymbol{n}}^{(3)} = \pm\,[-0.383\ \ -0.924\ 0]^{\mathrm{T}}$$

Problem 16 Find eigenstresses and eigenvectors of the stress tensor with components:

$$\begin{bmatrix} 200 & -\sqrt{3}\cdot 100 & 0 \\ -\sqrt{3}\cdot 100 & 400 & 0 \\ 0 & 0 & 400 \end{bmatrix} \quad \text{MPa}$$

Answers:

$$\lambda^{(1)} = 100 \quad \lambda^{(2)} = 400 \quad \lambda^{(3)} = 500 \quad \text{MPa}$$

$$\hat{\boldsymbol{n}}^{(1)} = \pm \left[\frac{\sqrt{3}}{2}\ \frac{1}{2}\ 0\right]^{\mathrm{T}} \quad \hat{\boldsymbol{n}}^{(2)} = \pm\,[0\ 0\ 1]^{\mathrm{T}} \quad \hat{\boldsymbol{n}}^{(3)} = \pm \left[\frac{1}{2}\ -\frac{\sqrt{3}}{2}\ 0\right]^{\mathrm{T}}$$

Matlab example 5 An example of using Matlab commands for matrix definitions (for A) and computing the eigenvalues and eigenvectors given below:

```
>> A=[1 2 3; 2 4 5; 3 5 6];
>> [n,lambda]=eig(A)

n =

    0.7370    0.5910    0.3280
    0.3280   -0.7370    0.5910
   -0.5910    0.3280    0.7370


lambda =

   -0.5157         0         0
         0    0.1709         0
         0         0   11.3448
```

Python example 5 An example of using Python commands for matrix definitions (for A) and computing the eigenvalues and eigenvectors given below:

```
import numpy as np
import numpy.linalg as la
A = np.array([[1.,2.,3.],[2.,4.,5.],[3.,5.,6.]])
```

```
eigenvalues, eigenvectors=la.eig(A)
print(eigenvalues)
print(eigenvectors)

[11.34481428 -0.51572947  0.17091519]
[[-0.32798528 -0.73697623  0.59100905]
 [-0.59100905 -0.32798528 -0.73697623]
 [-0.73697623  0.59100905  0.32798528]]
```

The proof that I_1, I_2 and I_3 are invariant with respect to coordinate system follows from showing that they will remain the same if expressed in components of another coordinate system T'_{ij}. By using the coordinate transformation $T_{ij} = l^T_{ik}\, T'_{kl}\, l_{lj}$ and the orthogonal property of the coordinate transformation matrix l_{ij} we can rewrite I_1 as:

$$I_1(T_{ij}) = T_{ii} = l^T_{ik}\, T'_{kl}\, l_{li} = \underbrace{l_{li}\, l^T_{ik}}_{\delta_{lk}}\, T'_{kl} = T'_{kk} = I_1(T'_{ij}) \tag{1.50}$$

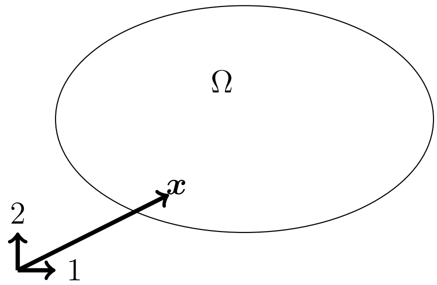
The second invariant I_2 consists of I_1 and $T_{ij}\, T_{ij}$ that can be rewritten as:

$$T_{ij}\, T_{ij} = l^T_{ik}\, T'_{kl}\, l_{lj}\, l^T_{im}\, T'_{mn}\, l_{nj} = \underbrace{l_{ki}\, l^T_{im}}_{\delta_{km}}\, \underbrace{l_{nj}\, l^T_{jl}}_{\delta_{nl}}\, T'_{kl}\, T'_{mn} = T'_{kl}\, T'_{kl} \tag{1.51}$$

whereby $I_2(T_{ij}) = I_2(T'_{ij})$. To show that I_3 also is invariant we first note that since $l_{li}\, l^T_{ik} = \delta_{lk}$ then $\det(l_{li}\, l^T_{ik}) = \underbrace{\det(l_{li})\det(l^T_{ik})}_{\text{no sum on } i} = 1$. Now we use this in I_3:

$$I_3(T_{ij}) = \det(T_{ij}) = \det(l^T_{ik}\, T'_{kl}\, l_{lj}) = \det(l^T_{ik})\det(T'_{kl})\det(l_{lj}) = \det(T'_{kl}) = I_3(T'_{kl}) \tag{1.52}$$

1.5 Spatial derivatives



A tensor field describes how the tensor depends on the spatial location $\boldsymbol{x}$ in the body Ω and the time t, e.g.

- Scalar field (such as temperature, pressure) $\phi = \phi(\boldsymbol{x}, t)$ or $\phi = \phi(x_i, t)$
- Vector field (such as displacement, velocity, force) $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ or $u_i = u_i(x_j, t)$

- Second-order tensor field (such as stress, strain) $\boldsymbol{T} = \boldsymbol{T}(\boldsymbol{x}, t)$ or $T_{ij} = T_{ij}(x_k, t)$.

To measure how such quantities change within the body the gradient (differential vector) operator $\boldsymbol{\nabla}$ is introduced as

$$\boldsymbol{\nabla} = \hat{\boldsymbol{e}}_i \frac{\partial}{\partial x_i} \tag{1.53}$$

By applying the gradient operator via an open product (from the left) to a scalar field $\phi(\boldsymbol{x}, t)$, a vector field $\boldsymbol{u}(\boldsymbol{x}, t)$ and a second-order tensor field $\boldsymbol{T}(\boldsymbol{x}, t)$ the following results are obtained

$$\boldsymbol{\nabla}\phi = \frac{\partial \phi}{\partial x_i}\hat{\boldsymbol{e}}_i\ , \quad \boldsymbol{\nabla}\boldsymbol{u} = \frac{\partial u_j}{\partial x_i}\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_j\ , \quad \boldsymbol{\nabla}\boldsymbol{T} = \frac{\partial T_{jk}}{\partial x_i}\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_j\hat{\boldsymbol{e}}_k. \tag{1.54}$$

It can be noted that the tensor fields are always increased by one degree in this using this procedure. Later we will also need to apply ∇ from the right on a vector field $\boldsymbol{u}(\boldsymbol{x}, t)$ which is defined as

$$\boldsymbol{u}\nabla = \frac{\partial u_i}{\partial x_j}\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j \tag{1.55}$$

By instead applying the gradient operator via a scalar product (from the left) to a vector field $\boldsymbol{u}(\boldsymbol{x}, t)$ and a second-order tensor field $\boldsymbol{T}(\boldsymbol{x}, t)$ result in

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = \frac{\partial u_i}{\partial x_i}\ , \quad \boldsymbol{\nabla}\cdot\boldsymbol{T} = \frac{\partial T_{ij}}{\partial x_i}\hat{\boldsymbol{e}}_j. \tag{1.56}$$

This is also called the divergence with the following notation

$$\mathrm{div}(\boldsymbol{u}) = \boldsymbol{\nabla}\cdot\boldsymbol{u}\ , \quad \mathrm{div}(\boldsymbol{T}) = \boldsymbol{\nabla}\cdot\boldsymbol{T}. \tag{1.57}$$

For the divergence operator the tensor fields are always decreased by one degree.

Another product that can be used with the gradient operator is the vector product. The vector product with the gradient operator defines the curl of a vector field

$$\mathrm{curl}(\boldsymbol{u}) = \boldsymbol{\nabla}\times\boldsymbol{u} = e_{ijk}\partial_i u_j \hat{\boldsymbol{e}}_k \tag{1.58}$$

To further compress the notation we introduce the index form of the gradient operator, $\partial_j = \partial/\partial x_j = \hat{\boldsymbol{e}}_j \cdot \boldsymbol{\nabla}$, or even more compactly, a subscripted comma which for example results in:

$$\boldsymbol{\nabla}\boldsymbol{u} = \partial_i u_j\,\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_j = u_{j,i}\,\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_j\ , \quad \mathrm{div}(\boldsymbol{T}) = \partial_i T_{ij}\,\hat{\boldsymbol{e}}_j = T_{ij,i}\,\hat{\boldsymbol{e}}_j. \tag{1.59}$$

Later in this text we will omit the base vectors and simply work with the components of the tensors e.g. $\partial u_i/\partial x_j$, $u_{i,j}$, $T_{ij,j}$ etc.

■ **Example 1.7** a) The temperature varies in the coordinate system $\{\hat{\boldsymbol{e}}_i\}$ with coordinates x_1, x_2 as $\Phi(x_1, x_2) = x_1^2 + (x_2/\alpha)^2 + \beta = 0$. The temperature gradient becomes:

$$\nabla\Phi = 2\,x_1\,\hat{\boldsymbol{e}}_1 + 2\,x_2/\alpha^2\,\hat{\boldsymbol{e}}_2$$

b) Assume that the displacement field is $\boldsymbol{u}(\boldsymbol{x}) = x_2\,\hat{\boldsymbol{e}}_1 + L\,\mathrm{e}^{(-x_1-x_2)/a}\,\hat{\boldsymbol{e}}_2$. The strain $\boldsymbol{\epsilon}$ is defined as $\boldsymbol{\epsilon} = (\nabla\,\boldsymbol{u} + \boldsymbol{u}\,\nabla)/2$ is obtained as

$$\boldsymbol{\epsilon} = 1/2\,(u_{j,i} + u_{i,j})\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j = 1/2\,(-L/a\,\mathrm{e}^{(-x_1-x_2)/a} + 1)\hat{\boldsymbol{e}}_1\,\hat{\boldsymbol{e}}_2 + 1/2\,(1 - L/a\,\mathrm{e}^{(-x_1-x_2)/a})\hat{\boldsymbol{e}}_2\,\hat{\boldsymbol{e}}_1 - L/a\,\mathrm{e}^{(-x_1-x_2)/a}\,\hat{\boldsymbol{e}}_2\,\hat{\boldsymbol{e}}_2$$

c) For the displacement field in b) the divergence of $\boldsymbol{u}$ becomes:

$$\nabla\cdot\boldsymbol{u} = -L/a\,\mathrm{e}^{(-x_1-x_2)/a}$$

d) For the displacement field in b) the curl of $\boldsymbol{u}$ becomes:

$$\nabla\times\boldsymbol{u} = e_{123}\,u_{2,1}\hat{\boldsymbol{e}}_3 + e_{213}\,u_{1,2}\hat{\boldsymbol{e}}_3 = (-L/a\,\mathrm{e}^{(-x_1-x_2)/a} - 1)\,\hat{\boldsymbol{e}}_3$$

■

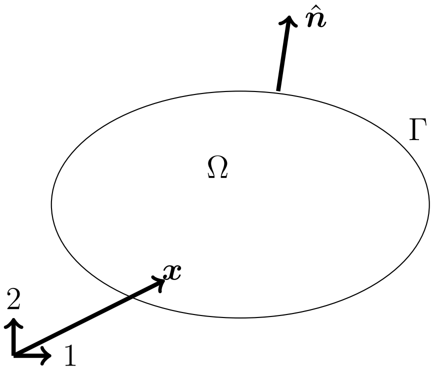
Problem 17 Show that:

a) $\nabla(\boldsymbol{a}\cdot\boldsymbol{x}) = \boldsymbol{a} + \nabla\boldsymbol{a}\cdot\boldsymbol{x}$

b) $\nabla\cdot(\boldsymbol{a}\times\boldsymbol{b}) = (\nabla\times\boldsymbol{a})\cdot\boldsymbol{b} - (\nabla\times\boldsymbol{b})\cdot\boldsymbol{a}$

c) $\nabla\cdot(\boldsymbol{A}\cdot\boldsymbol{b}) = (\nabla\cdot\boldsymbol{A})\cdot\boldsymbol{b} + \boldsymbol{A}:\nabla\boldsymbol{b}$

1.6 Divergence theorem



Gauss' divergence theorem is an important and useful theorem, which allows us to convert the volume integral of a divergence into a surface integral as follows

$$\int_\Omega \boldsymbol{\nabla}\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{x} = \oint_\Gamma \hat{\boldsymbol{n}}\cdot\boldsymbol{u}\,\mathrm{d}s \quad \text{or} \quad \int_\Omega u_{i,i}\,\mathrm{d}\boldsymbol{x} = \oint_\Gamma \hat{n}_i u_i\,\mathrm{d}s \tag{1.60}$$

where Γ is the closed boundary surface of Ω, and $\hat{\boldsymbol{n}}$ is the outward normal unit vector to Γ. This theorem can now be applied for tensor fields $\boldsymbol{T}$ by setting $u_i = T_{i1}$, T_{i2} and T_{i3}. Thereby we obtain

$$\begin{cases} \int_\Omega T_{i1,i}\,\mathrm{d}\boldsymbol{x} &= \oint_\Gamma \hat{n}_i T_{i1}\,\mathrm{d}s \\ \int_\Omega T_{i2,i}\,\mathrm{d}\boldsymbol{x} &= \oint_\Gamma \hat{n}_i T_{i2}\,\mathrm{d}s \\ \int_\Omega T_{i3,i}\,\mathrm{d}\boldsymbol{x} &= \oint_\Gamma \hat{n}_i T_{i3}\,\mathrm{d}s \end{cases}$$

which can be summarized as:

$$\int_{\Omega} T_{ij,i}\,\mathrm{d}\boldsymbol{x} = \oint_{\Gamma} \hat{n}_i T_{ij}\,\mathrm{d}s \quad \text{or} \quad \int_{\Omega} \mathrm{div}(\boldsymbol{T})\,\mathrm{d}\boldsymbol{x} = \oint_{\Gamma} \hat{\boldsymbol{n}}\cdot\boldsymbol{T}\,\mathrm{d}s \tag{1.61}$$

If we instead apply the Gauss' divergence theorem to a scalar field for the three cases

$$\begin{cases} u_1 & = \phi,\ u_2 = u_3 = 0 \\ u_1 & = 0,\ u_2 = \phi,\ u_3 = 0 \\ u_1 & = 0,\ u_2 = 0,\ u_3 = \phi \end{cases}$$

we obtain

$$\int_{\Omega} \partial\phi_{,i}\,\mathrm{d}\boldsymbol{x} = \oint_{\Gamma} \hat{n}_i\phi\,\mathrm{d}s \quad \text{or} \quad \int_{\Omega} \nabla\phi\,\mathrm{d}\boldsymbol{x} = \oint_{\Gamma} \hat{\boldsymbol{n}}\phi\,\mathrm{d}s \tag{1.62}$$

In practice, the name divergence theorem refers to equations (1.60), (1.61) and (1.62).

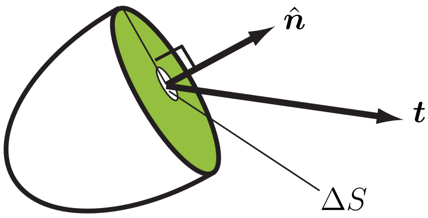
Problem 18 By using the divergence theorem show that: $\oint_{\Gamma} x_i\,\hat{n}_j\,\mathrm{d}s = V\,\delta_{ij}$.

2. Stress, motion and deformation

2.1 Stress analysis

The stress (also called traction) vector $\boldsymbol{t}(\hat{\boldsymbol{n}})$ is defined as the force acting on an area with normal $\hat{\boldsymbol{n}}$. In a point of a body the stress vector is defined as

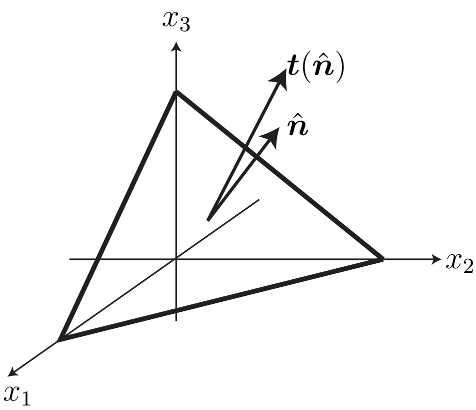
$$\boldsymbol{t}(\hat{\boldsymbol{n}}) = \lim_{\Delta S \to 0} \frac{\Delta \boldsymbol{F}}{\Delta S} \tag{2.1}$$



A property of the stress vector is that it must follow Newton's third law for action and reaction. Therefore, in the same point of a body the stress vector on the area with normal $\hat{\boldsymbol{n}}$ and normal $-\hat{\boldsymbol{n}}$ must be opposite. This means that

$$\boldsymbol{t}(\hat{\boldsymbol{n}}) = -\boldsymbol{t}(-\hat{\boldsymbol{n}}). \tag{2.2}$$

To find a relation between the normal $\hat{\boldsymbol{n}}$ and the stress vector $\boldsymbol{t}(\hat{\boldsymbol{n}})$ we study a tetrahedral element:



The tetrahedron is assumed to have the four surfaces defined as

1. normal $\hat{\boldsymbol{n}}$ and area A subjected to stress vector $\boldsymbol{t}(\hat{\boldsymbol{n}})$,

2. normal $-\hat{\boldsymbol{e}}_1$ and area A_1 subjected to stress vector $\boldsymbol{t}(-\hat{\boldsymbol{e}}_1)$,
3. normal $-\hat{\boldsymbol{e}}_2$ and area A_2 subjected to stress vector $\boldsymbol{t}(-\hat{\boldsymbol{e}}_2)$,
4. normal $-\hat{\boldsymbol{e}}_3$ and area A_3 subjected to stress vector $\boldsymbol{t}(-\hat{\boldsymbol{e}}_3)$.

The relation between the areas A, A_1, A_2, A_3 and the components of $\hat{\boldsymbol{n}}$ can be derived via the divergence theorem as $\hat{n}_i = A_i/A$, see the following example.

■ **Example 2.1** Choose $\phi = 1$ and apply the divergence theorem according to (1.62) to the tetrahedron

$$\oint_\Gamma \hat{\boldsymbol{n}}\phi \,\mathrm{d}s = \oint_\Gamma \hat{\boldsymbol{n}} \,\mathrm{d}s = -\hat{\boldsymbol{e}}_1 A_1 - \hat{\boldsymbol{e}}_2 A_2 - \hat{\boldsymbol{e}}_3 A_3 + \hat{\boldsymbol{n}} A = \int_\Omega \nabla\phi = 0$$

From these three equations we can identify that $A_i = \hat{n}_i A$. ■

The next step is now to study equilibrium of the tetrahedron:

$$\boldsymbol{t}(\hat{\boldsymbol{n}}) A + \boldsymbol{t}(-\hat{\boldsymbol{e}}_1) A_1 + \boldsymbol{t}(-\hat{\boldsymbol{e}}_2) A_2 + \boldsymbol{t}(-\hat{\boldsymbol{e}}_3) A_3 = \mathbf{0}.$$

If we use the relation between the areas and Newton's third law we obtain:

$$\boldsymbol{t}(\hat{\boldsymbol{n}}) = \boldsymbol{t}(\hat{\boldsymbol{e}}_1)\hat{n}_1 + \boldsymbol{t}(\hat{\boldsymbol{e}}_2)\hat{n}_2 + \boldsymbol{t}(\hat{\boldsymbol{e}}_3)\hat{n}_3. \tag{2.3}$$

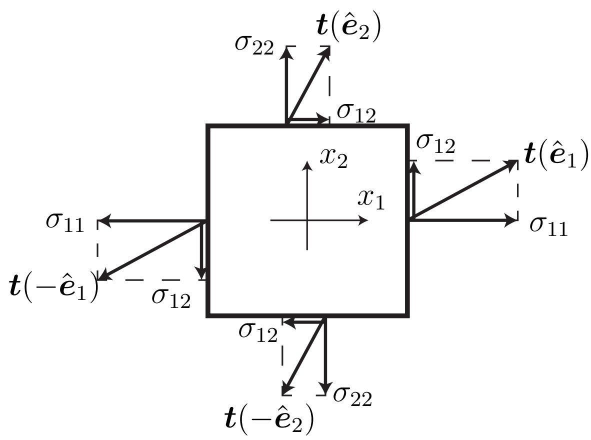
The second-order stress tensor $\boldsymbol{\sigma}$ is defined based on $\boldsymbol{t}(\hat{\boldsymbol{e}}_i)$ such that

$$[\sigma_{ij}] = [t_j(\hat{\boldsymbol{e}}_i)] \tag{2.4}$$

or more explicitly

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} t_1(\hat{\boldsymbol{e}}_1) & t_2(\hat{\boldsymbol{e}}_1) & t_3(\hat{\boldsymbol{e}}_1) \\ t_1(\hat{\boldsymbol{e}}_2) & t_2(\hat{\boldsymbol{e}}_2) & t_3(\hat{\boldsymbol{e}}_2) \\ t_1(\hat{\boldsymbol{e}}_3) & t_2(\hat{\boldsymbol{e}}_3) & t_3(\hat{\boldsymbol{e}}_3) \end{bmatrix} \tag{2.5}$$

This can be graphically shown as (here 2d):



To sum up, the relation between the stress vector $\boldsymbol{t}$ and normal vector $\hat{\boldsymbol{n}}$ is obtained via the stress tensor $\boldsymbol{\sigma}$ as follows:

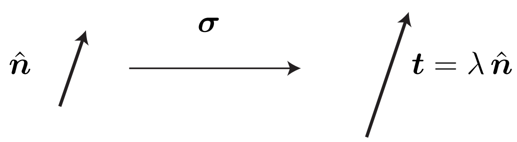
$$\boldsymbol{t} = \boldsymbol{n} \cdot \boldsymbol{\sigma} = \boldsymbol{\sigma}^T \cdot \boldsymbol{n} \quad \text{or } t_i = n_j\, \sigma_{ji} = \sigma^T_{ij}\, n_j. \tag{2.6}$$

This relation is the so-called Cauchy's formula. As will be proven later in the course, the stress tensor is symmetric due to principle of angular momentum i.e. $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$ and, hence, this relation can be written as $\boldsymbol{t} = \boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}}$. We can conclude that if the stress tensor $\boldsymbol{\sigma}$ is known in a point of the body then it is possible to compute the stress vector $\boldsymbol{t}$ on any plane through the point. This is called the Cauchy's stress principle.

Often the components of the stress tensor are divided into normal stresses and shear stresses. The normal stresses are the diagonal components of the stress tensor i.e. σ_{11}, σ_{22} and σ_{33} whereas the shear stresses are the off-diagonal components i.e. σ_{12}, σ_{23} and σ_{13}. Note that the terminology normal and shear components relate to what plane that is chosen. In the figure above the choice of plane is defined by the normal $\hat{\boldsymbol{e}}_1$ or $\hat{\boldsymbol{e}}_2$. In general, the normal component of the stress on a plane with normal $\hat{\boldsymbol{n}}$ is obtained from

$$\sigma_{nn} = \hat{\boldsymbol{n}} \cdot \boldsymbol{t} = \hat{\boldsymbol{n}} \cdot \boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}} = \boldsymbol{\sigma} : (\hat{\boldsymbol{n}}\, \hat{\boldsymbol{n}}) = \sigma_{ij}\, \hat{n}_i\, \hat{n}_j. \tag{2.7}$$

Let us now adopt the concept of eigenvalues and eigenvectors for a stress tensor $\boldsymbol{\sigma}$. The eigenvector is a direction $\hat{\boldsymbol{n}}$ that is not changed upon a scalar multiplication with the stress tensor $\boldsymbol{\sigma}$:



This means that on a plane with the normal being an eigenvector of $\boldsymbol{\sigma}$ then the stress vector $\boldsymbol{t}$ is parallel to the normal i.e. $\boldsymbol{t} = \lambda\, \hat{\boldsymbol{n}}$. In other words, on such a plane only the normal components are non-zero.

Often the stress tensor $\boldsymbol{\sigma}$ is additatively decomposed into a deviatoric $\boldsymbol{\sigma}^{\text{dev}}$ and a spherical (hydrostatic) tensor $\sigma_{\text{m}}\, \boldsymbol{\delta}$ as follows:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\text{dev}} + \sigma_{\text{m}}\, \boldsymbol{\delta} \quad \text{or } \sigma_{ij} = \sigma^{\text{dev}}_{ij} + \sigma_{\text{m}}\, \delta_{ij} \tag{2.8}$$

with

$$\sigma_{\text{m}} = \sigma_{kk}/3 \quad \text{and } \sigma^{\text{dev}}_{ij} = \sigma_{ij} - \sigma_{\text{m}}\, \delta_{ij}. \tag{2.9}$$

Problem 19 Assume that the stress tensor field $\boldsymbol{\sigma}$ is represented in the coordinate system $\hat{\boldsymbol{e}}_i$ with the following components

$$[\sigma_{ij}] = \begin{bmatrix} 100 & 50 & 200\,x_2 \\ 50 & 100 & 400\,x_1 \\ 200\,x_2 & 400\,x_1 & 100 \end{bmatrix}$$

where the stress components are in [MPa] and the coordinates are in [mm]. At the point $\boldsymbol{x} = \hat{\boldsymbol{e}}_1 + 2\,\hat{\boldsymbol{e}}_2 + 3\,\hat{\boldsymbol{e}}_3$ [mm] and on the plane with normal $\hat{\boldsymbol{n}} = (\hat{\boldsymbol{e}}_1 - \hat{\boldsymbol{e}}_2 + \hat{\boldsymbol{e}}_3)/\sqrt{3}$

a) Determine the stress vector $\boldsymbol{t}$

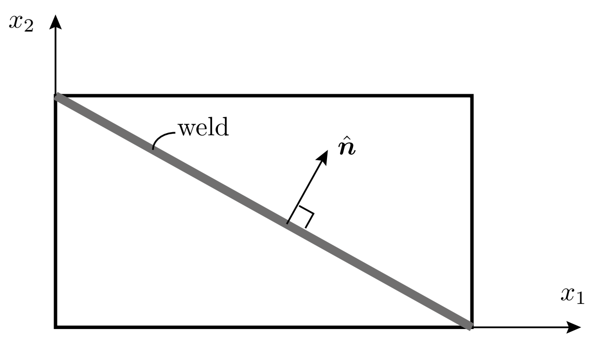
b) Determine the normal and shear components of $\boldsymbol{t}$.

Answers:

a)

$$[\boldsymbol{t}] = \frac{1}{\sqrt{3}} \begin{bmatrix} 450 \\ 350 \\ 100 \end{bmatrix} \quad \text{MPa}$$

b) $\sigma_{nn} \approx 66.7$ MPa, $t_s \approx 327$ MPa.



Problem 20

A welded structure is subjected to a homogeneous plane stress condition described in a Cartesian coordinate system 123 as:

$$[\sigma_{ij}] = \begin{bmatrix} \bar{\sigma} & \bar{\sigma}/3 & 0 \\ \bar{\sigma}/3 & \bar{\sigma} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The direction of the weld is given by the normal $[\hat{\boldsymbol{n}}] = 1/\sqrt{5}\ [1\ 2\ 0]^{\mathrm{T}}$ (see figure).

a) Determine the stress (traction) vector $\boldsymbol{t}$ acting on the weld expressed in $\bar{\sigma}$.

b) Assume that the largest allowable shear stress in the weld is 200 [MPa]. What is the largest allowable value of $\bar{\sigma}$?

Answers

a) $[\boldsymbol{t}] = \bar{\sigma}/(3\sqrt{5})\ [5\ 7\ 0]^{\mathrm{T}}$
b) $\bar{\sigma} \leq 1000$ MPa.

2.2 Continuum motion

The motion of a continuum (material volume) is shown in Figure 2.1. A material particle P may be identified by its initial (or reference) position $\boldsymbol{X}$. The current position $\boldsymbol{x}$, of a material particle is then defined by a function

$$x_i = x_i\,(\boldsymbol{X}, t) \tag{2.10}$$

The displacement $\boldsymbol{u}$ of a particle P is defined as

$$u_i = x_i - X_i \tag{2.11}$$

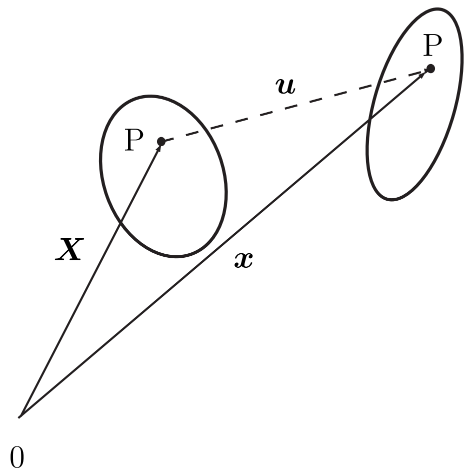


Figure 2.1: Illustration of motion of a continuum.

A key quantity that describes the deformation of the body (material volume) is the *deformation gradient* $\boldsymbol{F}$. The deformation gradient describes the relation between a line element $\mathrm{d}\boldsymbol{X}$ at the material particle P in the initial (undeformed) body and the corresponding line element $\mathrm{d}\boldsymbol{x}$ at the material particle P in the current (undeformed) body, i.e.

$$\mathrm{d}\boldsymbol{x} = \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{X} \quad\text{or}\quad F_{ij} = \frac{\partial\, x_i}{\partial\, X_j} = \delta_{ij} + \frac{\partial\, u_i}{\partial\, X_j} \quad\text{or}\quad \boldsymbol{F} = \boldsymbol{x}\nabla_0 \tag{2.12}$$

which is also illustrated in Figure 2.2. Based on the deformation gradient $\boldsymbol{F}$ a number of strain measures can be defined. An example is the frequently used Green-Lagrange strain

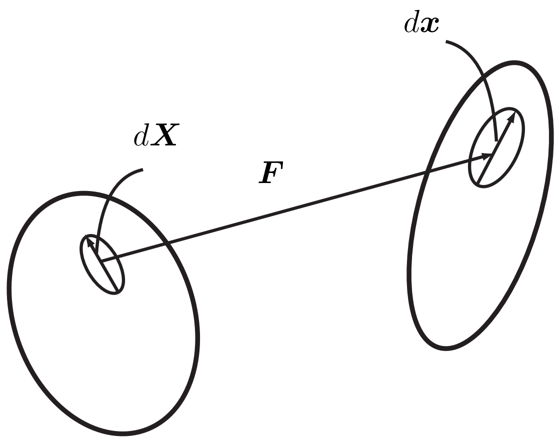


Figure 2.2: Illustration of deformation gradient.

$\boldsymbol{E}$ defined as follows:

$$\boldsymbol{E} = \frac{1}{2}\left(\boldsymbol{F}^{\mathrm{T}} \cdot \boldsymbol{F} - \boldsymbol{\delta}\right) \quad \text{or} \tag{2.13}$$

$$E_{ij} = \frac{1}{2}\left(F_{ik}^{\mathrm{T}} F_{kj} - \delta_{ij}\right) = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right) \tag{2.14}$$

For the special case of small deformations $\boldsymbol{E}$ approaches the usual small strain tensor $\boldsymbol{\epsilon}$, i.e.

$$E_{ij} \approx \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) \approx \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \epsilon_{ij} \tag{2.15}$$

We can note that both the Green-Lagrange strain $\boldsymbol{E}$ and the small strain tensor $\boldsymbol{\epsilon}$ are symmetric, i.e. $\boldsymbol{E}^{\mathrm{T}} = \boldsymbol{E}$ and $\boldsymbol{\epsilon}^{\mathrm{T}} = \boldsymbol{\epsilon}$.

2.3 Lagrangian and Eulerian description

Physical field quantities can be described in either a *Lagrangian* (or sometimes called material) or *Eulerian* description:

- Lagrangian description of scalars, vectors and second-order tensors:

$$\phi = \phi(\boldsymbol{X}, t)\ , \quad \boldsymbol{u} = \boldsymbol{u}(\boldsymbol{X}, t)\ , \quad \boldsymbol{T} = \boldsymbol{T}(\boldsymbol{X}, t) \quad \text{or}$$
$$\phi = \phi(X_i, t)\ , \quad u_i = u_i(X_j, t)\ , \quad T_{ij} = T_{ij}(X_k, t)$$

- Eulerian description of scalars, vectors and second-order tensors:

$$\phi = \phi(\boldsymbol{x}, t)\ , \quad \boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)\ , \quad \boldsymbol{T} = \boldsymbol{T}(\boldsymbol{x}, t) \quad \text{or}$$
$$\phi = \phi(x_i, t)\ , \quad u_i = u_i(x_j, t)\ , \quad T_{ij} = T_{ij}(x_k, t)$$

An important field quantity is the velocity $\boldsymbol{v}$ of a material particle P. The velocity is defined as the time derivative of the position vector $\boldsymbol{x}$:

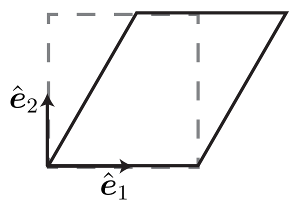
$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{x}(\boldsymbol{X}, t)}{\mathrm{d}t} \quad \text{or} \quad v_i = \frac{\mathrm{d}x_i(X_j, t)}{\mathrm{d}t} \tag{2.16}$$

whereby the velocity is described in an Lagrangian description $\boldsymbol{v}(\boldsymbol{X}, t)$. By assuming that the initial position of the particle $\boldsymbol{X}$ can be expressed in terms of $\boldsymbol{x}$ and t we can write the velocity in Eulerian description

$$\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x}, t) \tag{2.17}$$

Next follows three examples to illustrate the introduced concepts regarding motion.

■ **Example 2.2** Simple shear of a quadratic disc (side length h_0) where the upper boundary moves horizontally with velocity v_0:



The motion can be expressed as:

$$\begin{cases} x_1(X_1, X_2, X_3, t) &= X_1 + X_2\, v_0\, t/h_0 \\ x_2(X_1, X_2, X_3, t) &= X_2 \\ x_3(X_1, X_2, X_3, t) &= X_3 \end{cases}$$

whereby the velocity $\boldsymbol{v}$ can be obtained, in Lagrangian description, as:

$$v_i = \begin{bmatrix} X_2\, v_0/h_0 \\ 0 \\ 0 \end{bmatrix}$$

By using the expression for the motion the velocity can be written in an Eulerian description as

$$v_i = \begin{bmatrix} x_2\, v_0/h_0 \\ 0 \\ 0 \end{bmatrix}$$

Based on the expression for the motion we can also obtain the deformation gradient $\boldsymbol{F}$ as

$$F_{ij} = \begin{bmatrix} 1 & v_0\, t/h_0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and the Green Lagrange strain $\boldsymbol{E}$

$$E_{ij} = \frac{1}{2}\left(F^{\mathrm{T}}_{ik}\, F_{kj} - \delta_{ij}\right) = \ldots = \frac{1}{2}\begin{bmatrix} 0 & v_0\, t/h_0 & 0 \\ v_0\, t/h_0 & (v_0\, t/h_0)^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The displacement vector $\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X}$ is given as:

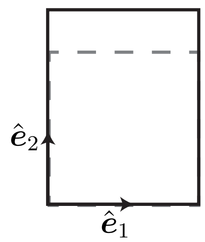
$$u_i = \begin{bmatrix} X_2\, v_0\, t/h_0 \\ 0 \\ 0 \end{bmatrix}$$

whereby the small strain tensor $\boldsymbol{\epsilon}$ becomes

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) = \begin{bmatrix} 0 & (v_0\,t/h_0)/2 & 0 \\ (v_0\,t/h_0)/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

■

■ **Example 2.3** Pure elongation of a quadratic disc (side length h_0) where the upper boundary moves vertically with velocity v_0:



The motion can be expressed as:

$$\begin{cases} x_1(X_1, X_2, X_3, t) &=& X_1 \\ x_2(X_1, X_2, X_3, t) &=& X_2 + X_2\,v_0\,t/h_0 \\ x_3(X_1, X_2, X_3, t) &=& X_3 \end{cases}$$

whereby the velocity $\boldsymbol{v}$ can be obtained, in Lagrangian description, as:

$$v_i = \begin{bmatrix} 0 \\ X_2\,v_0/h_0 \\ 0 \end{bmatrix}$$

By using the expression for the motion the velocity can be written in an Eulerian description as

$$v_i = \begin{bmatrix} 0 \\ x_2\,v_0/(h_0 + v_0\,t) \\ 0 \end{bmatrix}$$

Based on the expression for the motion we can also obtain the deformation gradient $\boldsymbol{F}$ as

$$F_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 + v_0\,t/h_0 & \\ 0 & 0 & 1 \end{bmatrix}$$

and the Green Lagrange strain $\boldsymbol{E}$

$$E_{ij} = \frac{1}{2}\left(F_{ik}^{\mathrm{T}}\,F_{kj} - \delta_{ij}\right) = \ldots = \begin{bmatrix} 0 & 0 & 0 \\ 0 & (v_0\,t/h_0)^2/2 + v_0\,t/h_0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The displacement vector $\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X}$ is given as:

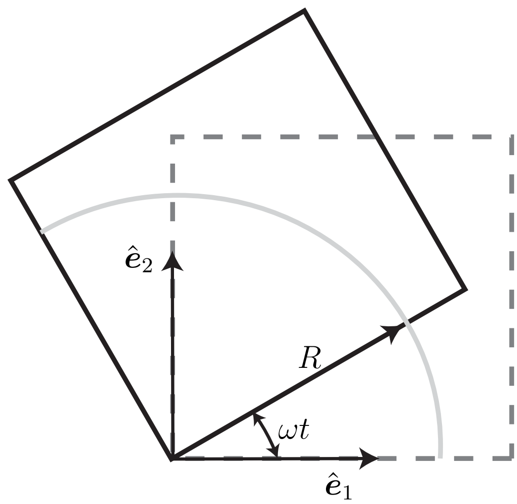
$$u_i = \begin{bmatrix} 0 \\ X_2\, v_0\, t/h_0 \\ 0 \end{bmatrix}$$

whereby the small strain tensor $\boldsymbol{\epsilon}$ becomes

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & v_0\, t/h_0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

■

■ **Example 2.4** Pure rotation around the left corner of a quadratic disc (side length h_0) with rotational velocity ω:



An arbitrary point's initial location in the disc is described by the distance $R = \sqrt{X_1^2 + X_2^2}$ to the left corner and angle $\alpha_0 = \text{atan}(X_2/X_1)$ (from the $\hat{\boldsymbol{e}}_1$ axis). During rotation the angle changes with rotation according to $\alpha = \alpha_0 + \omega t$ whereby the motion can be expressed as:

$$\begin{cases} x_1(X_1, X_2, X_3, t) &= R\cos(\alpha) \\ x_2(X_1, X_2, X_3, t) &= R\sin(\alpha) \\ x_3(X_1, X_2, X_3, t) &= X_3 \end{cases}$$

which can be (after some manipulations) written as

$$\begin{cases} x_1(X_1, X_2, X_3, t) &= X_1\cos(\omega t) - X_2\sin(\omega t) \\ x_2(X_1, X_2, X_3, t) &= X_1\sin(\omega t) + X_2\cos(\omega t) \\ x_3(X_1, X_2, X_3, t) &= X_3 \end{cases}$$

The velocity $\boldsymbol{v}$ can be obtained, in Lagrangian and Eulerian description, as :

$$v_i = \begin{bmatrix} \omega\,(-X_1\,\sin(\omega t) - X_2\,\cos(\omega t)) \\ \omega\,(X_1\,\cos(\omega t) - X_2\,\sin(\omega t)) \\ 0 \end{bmatrix} = \omega \begin{bmatrix} -x_2 \\ x_1 \\ 0 \end{bmatrix}$$

Based on the expression for the motion we can also obtain the deformation gradient $\boldsymbol{F}$ as

$$F_{ij} = \begin{bmatrix} \cos(\omega t) & -\sin(\omega t) & 0 \\ \sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and the Green Lagrange strain $\boldsymbol{E}$

$$E_{ij} = \frac{1}{2}\left(F^{\mathrm{T}}_{ik}\,F_{kj} - \delta_{ij}\right) = \ldots = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The displacement vector $\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X}$ is given as:

$$u_i = \begin{bmatrix} X_1\,(\cos(\omega t) - 1) - X_2\,\sin(\omega t) \\ \\ X_1\,\sin(\omega t) + X_2\,(\cos(\omega t) - 1) \\ 0 \end{bmatrix}$$

whereby the small strain tensor $\boldsymbol{\epsilon}$ becomes

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) = \begin{bmatrix} \cos(\omega t) - 1 & 0 & 0 \\ 0 & \cos(\omega t) - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

■

Problem 21 The motion of a body is given as

$$\boldsymbol{x}(\boldsymbol{X}, t) = X_1\,\hat{\boldsymbol{e}}_1 + (X_2 + X_1\,t/t_0)\,\hat{\boldsymbol{e}}_2 + (X_3 + X_2\,(1 - \mathrm{e}^{-t/t_0})\,\hat{\boldsymbol{e}}_3$$

where t_0 is a constant [s]. Determine

a) The deformation gradient $\boldsymbol{F}$

b) The Green-Lagrange strain $\boldsymbol{E}$

c) The velocity in Eulerian description $\boldsymbol{v}(\boldsymbol{x})$

Answers

$$\text{a)}\ [\boldsymbol{F}] = \begin{bmatrix} 1 & 0 & 0 \\ t/t_0 & 1 & 0 \\ 0 & (1 - \mathrm{e}^{-t/t_0}) & 1 \end{bmatrix}$$

$$\text{b)}\ [\boldsymbol{E}] = \begin{bmatrix} (t/t_0)^2/2 & (t/t_0)/2 & 0 \\ (t/t_0)/2 & 1/2\,(1-\mathrm{e}^{-t/t_0})^2 & 1/2\,(1-\mathrm{e}^{-t/t_0})) \\ 0 & 1/2\,(1-\mathrm{e}^{-t/t_0}) & 0 \end{bmatrix}$$

c)

$$[\boldsymbol{v}] = \begin{bmatrix} 0 & x_1/t_0 & 1/t_0\,(x_2 - x_1\,t/t_0)\,\mathrm{e}^{-t/t_0} \end{bmatrix}^{\mathrm{T}}$$

2.4 Material time derivative

Physical field quantities such as temperature, velocity, stress tensor change with time. This change is naturally described as the time derivative of the physical quantity of a material particle in the continuum. The particle is uniquely identified by the Lagrangian (material) vector $\boldsymbol{X}$. Therefore, it is useful to introduce the material time derivative which is denoted $\mathrm{D}(\bullet)/\mathrm{D}t$, $\dot{(\bullet)}$ or $\mathrm{d}(\bullet)/\mathrm{d}t$. If the physical quantity ϕ is described in an Lagrangian description $\phi(\boldsymbol{X}, t)$

$$\frac{\mathrm{D}\phi\,(\boldsymbol{X},t)}{\mathrm{D}t} = \dot{\phi}\,(\boldsymbol{X},t) = \frac{\mathrm{d}\phi\,(\boldsymbol{X},t)}{\mathrm{d}t}, \tag{2.18}$$

whereas the material time derivative of a field quantity described in an Eulerian description $\phi\,(\boldsymbol{x}, t)$ is obtained by the chain rule:

$$\begin{aligned} \dot{\phi}\,(\boldsymbol{x}(\boldsymbol{X},t),t) &= \frac{\mathrm{d}\phi\,(\boldsymbol{x}(\boldsymbol{X},t),t)}{\mathrm{d}t} \\ &= \frac{\partial\phi\,(\boldsymbol{x},t)}{\partial x_i}\frac{\partial x_i\,(\boldsymbol{X},t)}{\partial t} + \left.\frac{\partial\phi\,(\boldsymbol{x},t)}{\partial t}\right|_{\boldsymbol{x}} \end{aligned} \tag{2.19}$$

The first part in the result is the convective part while the second part is the time derivative of ϕ in a spatial position $\boldsymbol{x}$.

■ **Example 2.5** For Problem 21 the velocity $\boldsymbol{v}$ is given in Lagrangian and Eulerian coordinates as follows

$$[\boldsymbol{v}] = \begin{bmatrix} 0 \\ X_1/t_0 \\ 1/t_0\,X_2\,\mathrm{e}^{-t/t_0} \end{bmatrix} = \begin{bmatrix} 0 \\ x_1/t_0 \\ 1/t_0\,(x_2 - x_1\,t/t_0)\,\mathrm{e}^{-t/t_0} \end{bmatrix}$$

The acceleration can be obtained from the Lagrangian form directly as

$$[\boldsymbol{a}] = \begin{bmatrix} 0 \\ 0 \\ -1/t_0^2\,X_2\,\mathrm{e}^{-t/t_0} \end{bmatrix}$$

and the Eulerian form as

$$a_i = \frac{\partial v_i(\boldsymbol{x},t)}{\partial x_j}\frac{\partial x_j(\boldsymbol{X},t)}{\partial t} + \left.\frac{\partial v_i(\boldsymbol{x},t)}{\partial t}\right|_{\boldsymbol{x}}$$

which in matrix form becomes

$$[a_i] = \begin{bmatrix} 0 & 0 & 0 \\ 1/t_0 & 0 & 0 \\ -t/t_0^2\,\mathrm{e}^{-t/t_0} & 1/t_0\,\mathrm{e}^{-t/t_0} & 0 \end{bmatrix}\begin{bmatrix} 0 \\ x_1/t_0 \\ 1/t_0\,(x_2 - x_1\,t/t_0)\,\mathrm{e}^{-t/t_0} \end{bmatrix} + \\ + \begin{bmatrix} 0 \\ 0 \\ 1/t_0\,(-x_2/t_0 + x_1\,t/t_0^2 - x_1/t_0)\,\mathrm{e}^{-t/t_0} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1/t_0^2\,(x_2 - x_1\,t/t_0)\,\mathrm{e}^{-t/t_0} \end{bmatrix}$$

■

2.5 Reynolds' transport theorem for a material volume

In the balance laws physical quantities are integrated over the volume of interest. The integration can be performed for the current volume of the continuum Ω:

$$\int_\Omega \phi\,\mathrm{d}\boldsymbol{x}$$

By substituting the volume Ω to the initial volume (undeformed) of the continuum Ω_0 we obtain (using results from Math see e.g `https://en.wikipedia.org/wiki/Integration_by_substitution`):

$$\int_\Omega \phi\,\mathrm{d}\boldsymbol{x} = \int_{\Omega_0} \phi\,J\,\mathrm{d}\boldsymbol{X} \tag{2.20}$$

where $J = \det(\boldsymbol{F})$.

Note: The volume change of a body V/V_0 is given by $J = \det(\boldsymbol{F})$. This follows immediately from (2.20) by setting $\phi = 1$.

The material time derivative of a volume integral of ϕ can now be obtained as (using that Ω_0 is constant):

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega \phi\,\mathrm{d}\boldsymbol{x} = \frac{\mathrm{d}}{\mathrm{dt}}\int_{\Omega_0} \phi\,J\,\mathrm{d}\boldsymbol{X} = \int_{\Omega_0} \dot{\phi}\,J + \phi\,\dot{J}\,\mathrm{d}\boldsymbol{X} \tag{2.21}$$

The time derivative of the volume change $\dot{J}$ is given by (here without any proof):

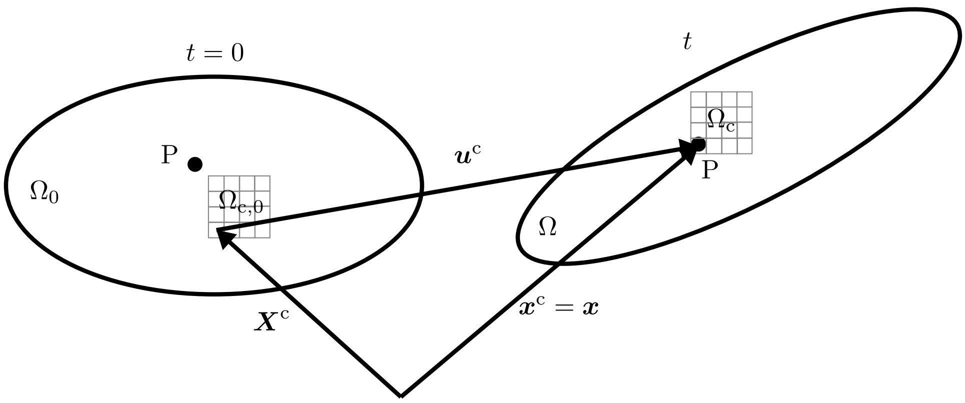
$$\dot{J} = \mathrm{div}(\boldsymbol{v})\,J \quad \text{or} \quad \dot{J} = v_{i,i}\,J \tag{2.22}$$

whereby we can obtain Reynold's transport theorem:

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega \phi\,\mathrm{d}\boldsymbol{x} = \int_\Omega \left(\frac{\mathrm{d}\phi}{\mathrm{dt}} + \phi\frac{\partial v_i}{\partial x_i}\right)\mathrm{d}\boldsymbol{x}. \tag{2.23}$$

2.6 Reynolds' transport theorem for a control volume

In fluid mechanics quantities are often measured in a control volume Ω_c with boundary $\Gamma_{\rm c}$. This control volume do not in general follow the movement of the material particles in the continuum as is illustrated in the figure below.



At time t the position of a control volume element element is $\boldsymbol{x}^{\rm c}$ and its velocity $\boldsymbol{v}^{\rm c}$. Note that although the position of a control volume element and a material point element P coincides at time t, i.e. $\boldsymbol{x}^{\rm c} = \boldsymbol{x}$, their velocities $\boldsymbol{v}^{\rm c}$ and $\boldsymbol{v}$ differ.

How $\int_{\Omega_{\rm c}} \phi \, {\rm d}\boldsymbol{x}$ changes with time can be obtained from the general form of Reynold's transport theorem:

$$\frac{\rm d}{\rm dt} \int_{\Omega_{\rm c}} \phi \, {\rm d}\boldsymbol{x} = \int_{\Omega_{\rm c}} \left(\frac{{\rm d}\phi}{\rm dt} + \phi \frac{\partial v_i^{\rm c}}{\partial x_i^{\rm c}} \right) {\rm d}\boldsymbol{x}. \tag{2.24}$$

This follows from the same arguments as in (2.21)-(2.23). Two special cases are common:

- The control volume elements are equal material point elements (they follow the deformation) then $\boldsymbol{v}^{\rm c} = \boldsymbol{v}$ and (2.23) is re-obtained. This assumption that the control volume elements are "nailed" to the material elements are often used in solid mechanics.
- The control volume elements are fixed in time which means that $\boldsymbol{v}^{\rm c} = \boldsymbol{0}$. This gives that if $\phi(x_i^{\rm c}, t)$ we obtain:

$$\frac{\rm d}{\rm dt} \int_{\Omega_{\rm c}} \phi \, {\rm d}\boldsymbol{x} = \int_{\Omega_{\rm c}} \frac{{\rm d}\phi}{\rm dt} \, {\rm d}\boldsymbol{x} = \int_{\Omega_{\rm c}} \left.\frac{\partial \phi}{\partial t}\right|_{\boldsymbol{x}^{\rm c}} + \frac{\partial \phi}{\partial x_i^{\rm c}} v_i^{\rm c} \, {\rm d}\boldsymbol{x} = \int_{\Omega_{\rm c}} \left.\frac{\partial \phi}{\partial t}\right|_{\boldsymbol{x}^{\rm c}} {\rm d}\boldsymbol{x}$$

3. Field equations

3.1 Physical quantities of a continuum

We consider a body occupying a region Ω at time t. The state of the body is assumed to be given by the quantities: mass M; momentum $\boldsymbol{P}$; angular momentum $\boldsymbol{N}$; kinetic energy K and internal energy U. Before giving the expressions of these quantities we remind us the corresponding expressions of $\boldsymbol{P}$, $\boldsymbol{N}$ and K are given for a point mass m as:

$$\begin{aligned}
\boldsymbol{P} &= m\,\boldsymbol{v} \quad \text{or } P_i = m\,v_i \\
\boldsymbol{N} &= m\,\boldsymbol{x} \times \boldsymbol{v} \quad \text{or } N_i = m\,e_{ijk}\,x_j\,v_k \\
K &= m\,|\boldsymbol{v}|^2/2 \quad \text{or } K = m\,v_i\,v_i/2
\end{aligned}$$

With this at hand and by assuming a density field $\rho(\boldsymbol{x}, t)$ and a velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$ then the M, $\boldsymbol{P}$, $\boldsymbol{N}$ and K for a body can be expressed as:

$$M = \int_\Omega \rho\,\mathrm{d}\boldsymbol{x}, \tag{3.1}$$

$$P_i = \int_\Omega \rho v_i\,\mathrm{d}\boldsymbol{x}, \tag{3.2}$$

$$N_i = \int_\Omega e_{ijk}\,x_j \rho v_k\,\mathrm{d}\boldsymbol{x}, \tag{3.3}$$

$$K = \int_\Omega \frac{1}{2}\rho v_i v_i\,\mathrm{d}\boldsymbol{x}. \tag{3.4}$$

In addition to these quantities the internal energy U is also introduced. U represents energy such as strain energy and thermal energy which together with the kinetic energy sums up to total energy of the body. Later U will be given more explicitly but at this stage we assume that the body has an internal energy density field $e(\boldsymbol{x}, t)$ such that:

$$U = \int_\Omega \rho e\,\mathrm{d}\boldsymbol{x}. \tag{3.5}$$

3.2 Input quantities

A schematic figure of a continuous body is given in Figure 3.1 with the field variables ρ, $\boldsymbol{v}$ and e.

Now we assume that the body is subjected input quantities that can change the state of the body, see Figure 3.1. The mechanical loading is given by a volume force $\boldsymbol{f}$ (force per unit mass) and a boundary load $\boldsymbol{t}$ (force per unit area). Thereby the total force $\boldsymbol{F}$ and

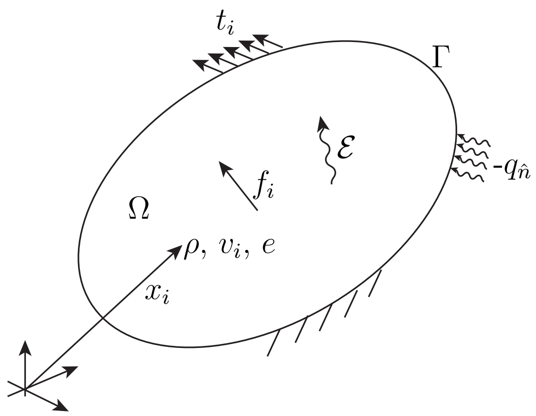


Figure 3.1: Illustration of a continuum Ω with boundary Γ.

moment $\boldsymbol{M}$ and mechanical power input to the body become:

$$F_i = \int_\Omega \rho f_i \,\mathrm{d}\boldsymbol{x} + \oint_\Gamma t_i \,\mathrm{d}s, \tag{3.6}$$

$$M_i = \int_\Omega e_{ijk} x_j \rho f_k \,\mathrm{d}\boldsymbol{x} + \oint_\Gamma e_{ijk} x_j t_k \,\mathrm{d}s., \tag{3.7}$$

$$\dot{W} = \int_\Omega \rho f_i v_i \,\mathrm{d}\boldsymbol{x} + \oint_\Gamma v_i t_i \,\mathrm{d}s, \tag{3.8}$$

Additionally, the body is subjected to the internal heat source $\mathcal{E}$ (energy per unit mass) and heat input $-q_{\hat{n}}$ (energy per unit area) resulting in the heat power input:

$$\dot{H} = \int_\Omega \rho \mathcal{E} \,\mathrm{d}\boldsymbol{x} + \oint_\Gamma -q_{\hat{n}} \,\mathrm{d}s \tag{3.9}$$

3.3 Physical conservation principles

Now the physical conservation principles are used to define how the state i.e. the mass M, the linear momentum $\boldsymbol{P}$, the angular momentum $\boldsymbol{N}$ and the total energy $K+U$ change of the body change with the mechanical and heat input.

3.3.1 Conservation of mass

Mass is in classical mechanics assumed to be conserved which can be written as:

$$\dot{M} = \frac{\mathrm{d}}{\mathrm{dt}} \int_\Omega \rho \,\mathrm{d}\boldsymbol{x} = 0 \tag{3.10}$$

By using Reynold's transport theorem (2.23) we obtain:

$$\dot{M} = \frac{\mathrm{d}}{\mathrm{dt}} \int_\Omega \rho \,\mathrm{d}\boldsymbol{x} = \int_\Omega \left(\dot{\rho} + \rho v_{i,i}\right) \,\mathrm{d}\boldsymbol{x} \tag{3.11}$$

The mass conservation is assumed for all choices of Ω (this argumentation is called localization) whereby:

$$\dot{\rho} + \rho\, v_{i,i} = 0 \quad in\ \Omega \tag{3.12}$$

This equation is called the *continuity equation*.

The continuity equation can be used together with Reynold's transport theorem to show that for an arbitrary field quantity ϕ:

$$\frac{\mathrm{d}}{\mathrm{dt}} \int_\Omega \rho\, \phi\, \mathrm{d}\boldsymbol{x} = \int_\Omega \frac{\mathrm{d}}{\mathrm{dt}}(\rho\, \phi) + \rho\, \phi\, v_{i,i}\, \mathrm{d}\boldsymbol{x} = \int_\Omega \rho\, \dot{\phi}\, \mathrm{d}\boldsymbol{x} \tag{3.13}$$

This result is denoted the modified Reynolds' transport theorem.[1]

Problem 22 Assume that the Eulerian description of the velocity field of a body is given as:

$$[v_i] = \begin{bmatrix} 2\, v_0\, x_1/L \\ v_0\, x_2/L \\ v_0\, x_3/L \end{bmatrix}$$

For the case that $v_0/L = 1$ [1/s] describe how the density ρ is changing from its initial value ρ_0.

Answer: $\rho = \rho_0\, \mathrm{e}^{-4t}$

3.3.2 Conservation of linear and angular momentum - Newton's laws

Newton's laws state that the material time derivative of the linear momentum $\boldsymbol{P}$ and the angular momentum $\boldsymbol{N}$ are determined by the applied force $\boldsymbol{F}$ and moment $\boldsymbol{M}$ as follows:

$$\dot{P}_i = F_i$$
$$\dot{N}_i = M_i$$

By using modified Reynold's tranport theorem (3.13) and equations (3.2) as well as (3.6) we obtain:

$$\dot{P}_i = \int_\Omega \rho\, \dot{v}_i\, \mathrm{d}\boldsymbol{x} = \int_\Omega \rho f_i\, \mathrm{d}\boldsymbol{x} + \oint_\Gamma t_i\, \mathrm{d}s \tag{3.14}$$

The next step is to use Cauchy's stress principle $t_i = \sigma_{ji}\hat{n}_j$ and the divergence theorem:

$$\int_\Omega \rho\, \dot{v}_i - \sigma_{ji,j} - \rho\, f_i\, \mathrm{d}\boldsymbol{x} = 0 \tag{3.15}$$

The localization argument now yields the momentum equation:

$$\sigma_{ji,j} + \rho\, f_i = \rho \dot{v}_i \tag{3.16}$$

[1]It can also be shown by instead integrating over the mass: $\frac{\mathrm{d}}{\mathrm{dt}} \int_\Omega \rho\, \phi\, \mathrm{d}\boldsymbol{x} = \frac{\mathrm{d}}{\mathrm{dt}} \int_m \phi\, \mathrm{dm} = \int_\mathrm{m} \dot{\phi}\, \mathrm{dm} = \int_\Omega \rho\, \dot{\phi}\, \mathrm{d}\boldsymbol{x}$

By using the same steps for the angular momentum $\dot{N}_i = M_i$ we will arrive at the result that the stress tensor must be symmetric:

$$\boldsymbol{\sigma}^{\mathrm{T}} = \boldsymbol{\sigma} \quad \text{or} \quad \sigma_{ij} = \sigma_{ji} \tag{3.17}$$

However, we leave those derivations of a Hand-in assignment.

Problem 23 For a plate with length L, height H and thickness b the non-zero stress field components are given by:

$$\sigma_{11} = -\frac{6\,f\,x_2}{H^2}(L^2 - x_1^2)\left(1 + \frac{2\,x_2^3/3 - H^2/10}{L^2 - x_1^2}\right),$$
$$\sigma_{22} = -f\,x_2\,(1 - 4\,x_2^2/H^2)/2, \quad \sigma_{12} = 3\,f\,x_1\,(1 - 4\,x_2^2/H^2)/2$$

The plate is subjected to quasistatic conditions, i.e. $\dot{v}_i \approx 0$, compute the volume force $[\mathrm{N/m^3}]$ that acts on the plate.

Answer:

$$[\rho\,f_i] = [0 \;\; -f \;\; 0]^{\mathrm{T}}$$

3.3.3 Conservation of energy - 1st law of thermodynamics

The 1st law of thermodynamics says that the material time derivative of the total energy of a body is equal to the power input:

$$\dot{K} + \dot{U} = \dot{W} + \dot{H} \tag{3.18}$$

If we now use modifed Reynold's transport theorem (3.8) and equations (3.4), (3.5),(3.8) as well as (3.9) we obtain:

$$\int_\Omega \rho v_i \dot{v}_i + \rho\,\dot{e}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \rho\,f_i\,v_i \mathrm{d}\boldsymbol{x} + \oint_\Gamma t_i\,v_i \mathrm{d}s + \int_\Omega \rho\mathcal{E}\,\mathrm{d}\boldsymbol{x} + \oint_\Gamma -q_{\hat{n}}\,\mathrm{d}s \tag{3.19}$$

Before we can use the localization argument the boundary integrals must be changed to volume integrals. The first of the boundary integrals is re-written using the Cauchy's stress theorem as follows:

$$\oint_\Gamma t_i\,v_i \mathrm{d}s = \oint_\Gamma \hat{n}_j \sigma_{ji}\,v_i \mathrm{d}\boldsymbol{x} = \int_\Omega \sigma_{ji,j}\,v_i + \sigma_{ji}\,v_{i,j} \mathrm{d}\boldsymbol{x} \tag{3.20}$$

The second boundary integral is re-written by assuming that the heat flux $q_{\hat{n}}$ is given by the heat flux vector $\boldsymbol{q}$ according to:

$$q_{\hat{n}} = \hat{\boldsymbol{n}} \cdot \boldsymbol{q} = \hat{n}_i\,q_i \tag{3.21}$$

thereby the divergence theorem gives us:

$$\oint_{\Gamma} -q_{\hat{n}}\,\mathrm{d}s = \int_{\Omega} -q_{i,i}\,\mathrm{d}\boldsymbol{x} \tag{3.22}$$

Now the 1st law of thermodynamics can be written as:

$$\int_{\Omega} \rho v_i \dot{v}_i + \rho\,\dot{e} - \rho\,f_i\,v_i - \sigma_{ji,j}\,v_i - \sigma_{ji}\,v_{i,j} - \rho\mathcal{E} + q_{i,i}\,\mathrm{d}\boldsymbol{x} = 0 \tag{3.23}$$

The balance of linear momentum (3.16) now together with the localization argument results in the energy equation:

$$\rho\dot{e} - v_{i,j}\sigma_{ij} + q_{i,i} = \rho\mathcal{E}. \tag{3.24}$$

3.4 Summary of field equations and field variables

A summary of the field equations and the field variables for a continuum are shown in the table below.

Balance law	Field variables	No equations
$\dot{\rho} + \rho\,v_{i,i} = 0$	$\rho,\ v_i$	1
$\rho\,\dot{v}_i = \sigma_{ji,j} + \rho\,f_i$	σ_{ij}	3
$\sigma_{ij} = \sigma_{ji}$	σ_{ij}	3
$\rho\,\dot{e} = \sigma_{ij}\,v_{i,j} - q_{i,i} + \rho\,\mathcal{E}$	$e,\ q_i$	1
	Tot. 17	Tot. 8

By counting the number of equations and unknowns we can conclude that 9 additional equations that must be formulated. These equations are the constitutive models that should mimic the material behaviour observed in experiments.

4. Constitutive models

The standard way to define constitutive models in solid mechanics, fluid mechanics and heat transfer is to describe how internal energy e, stress $\boldsymbol{\sigma}$ and heat flux $\boldsymbol{q}$ depend on other field variables such as density ρ, temperature θ, temperature gradient $\theta_{,i}$ displacement gradient $u_{i,j}$ and velocity gradient $v_{i,j}$. These models with their material parameters are based on experimental observations. In this section we will merely introduce some of most common (and simplest) constitutive models for heat transfer, fluids and solids.

The constitutive models are determined by the material properties. The material can be homogeneous meaning that properties are the same in the body Ω otherwise the material is heterogeneous. If the properties are the same in all directions then the material is called isotropic. For some materials the properties are anisotropic. Examples of the latter are: wood, composites and fibre reinforced concrete.

4.1 Fourier's law of thermal conductivity

Heat can be transfered by convection (motion of fluid), radiation (electromagnetics) and conduction (diffusion processes). For heat conduction the standard constitutive model is Fourier's law. For an isotropic material this law takes the form:

$$q_i = -k\,\Theta_{,i} \tag{4.1}$$

where the linear coefficient k is the thermal conductivity and Θ is the temperature. The temperature Θ is now assumed to give the internal energy e according to:

$$e = c_{\mathrm{p}}\,\Theta \tag{4.2}$$

where the constant c_p is the heat capacity of the material. If we consider a purely thermal problem (i.e. assuming $\sigma_{ij} = 0$) then the energy equation (3.24) reads as follows:

$$\rho\,\dot{e} = -q_{i,i} + \rho\,\mathcal{E}$$

By inserting (4.1) and (4.2) then we obtain the transient heat conduction equation:

$$\rho\,c_{\mathrm{p}}\,\dot{\Theta} = k\,\Theta_{,ii} + \rho\,\mathcal{E} \tag{4.3}$$

4.2 Viscous fluids

The simplest possible contitutive of a fluid is an ideal fluid. In this model the stress is assumed to be purely volumetric:

$$\sigma_{ij} = -p(\rho, \Theta)\, \delta_{ij} \tag{4.4}$$

This means that such a fluid cannot sustain shear stresses. The pressure p is assumed to follow the ideal gas law

$$p(\rho, \Theta) = \rho\, R\, \Theta / m_{\mathrm{g}} \tag{4.5}$$

where R is the gas constant, m_{g} is the mean molecular mass of the gas.

Most fluids are not "ideal" since they are a bit "sticky" and are able to sustain shear stresses. Therefore, a viscous stress $\boldsymbol{\tau}$ is introduced for and the stress $\boldsymbol{\sigma}$ is additively decomposed according to

$$\sigma_{ij} = -p(\rho, \Theta)\, \delta_{ij} + \tau_{ij} \tag{4.6}$$

For the case of an isotropic *Newtonian viscous fluid* we assume that the viscous stress $\boldsymbol{\tau}$ is linear in terms of the strain rate tensor $\boldsymbol{D}$ as follows

$$\tau_{ij} = \lambda^* \, D_{kk}\, \delta_{ij} + 2\, \mu^* D_{ij} \tag{4.7}$$

where the strain rate tensor $\boldsymbol{D}$ is defined from as the symmetric part of the velocity gradient:

$$D_{ij} = \frac{1}{2}\left(v_{i,j} + v_{j,i}\right). \tag{4.8}$$

In (4.7) the material parameters λ^* and the dynamic (shear) viscosity μ^* were introduced. Now the stress becomes:

$$\sigma_{ij} = -p(\rho, \Theta)\, \delta_{ij} + \lambda^* \delta_{ij} D_{kk} + 2\, \mu^* D_{ij} \tag{4.9}$$

The mechanical pressure $p_{\mathrm{mech}} = -\sigma_{mm}/3$ can be computed as:

$$p_{\mathrm{mech}} = -\frac{1}{3}\sigma_{mm} = p(\rho, \Theta) - \left(\lambda^* + \frac{2}{3}\, \mu^*\right) D_{kk} \tag{4.10}$$

If we introduce the Stoke's condition that $p_{\mathrm{mech}} = p(\rho, \Theta)$ then we can for a Newtonian viscous fluid obtain that

$$\sigma_{ij} = \sigma_{ij}^{\mathrm{dev}} + \frac{\sigma_{kk}}{3}\delta_{ij} = 2\, \mu^* D_{ij}^{\mathrm{dev}} - p(\rho, \Theta)\, \delta_{ij} \tag{4.11}$$

where $\boldsymbol{\sigma}^{\mathrm{dev}}$ and $\boldsymbol{D}^{\mathrm{dev}}$ are the deviatoric stress and deviatoric strain rate tensor, respectively. Stoke's condition means that the pressure in the fluid is strain rate independent.

The Navier-Stoke's equations are now simply obtained by inserting (4.11) into balance of linear momentum (3.16) together with the continuity equation.

Often for fluids one can assume that they are incompressible. From the conservation of mass and the incompressibility $\dot{\rho} = 0$ it follows from the continuity equation (3.12) that:

$$v_{i,i} = D_{ii} = 0$$

In this case $\boldsymbol{D}^{\text{dev}} = \boldsymbol{D}$ and, without using Stoke's condition, we obtain (4.11) from (4.9).

Problem 24 Assume the constitutive equation $\sigma_{ij} = (-p + \lambda\, D_{kk})\, \delta_{ij} + 2\, \mu D_{ij}$. Show that the equations of motion can be expressed in the velocity field as:

$$\rho\, \dot{v}_i = \rho\, f_i - p_{,i} + (\lambda + \mu)\, v_{j,ij} + \mu v_{i,jj}.$$

4.3 Linear elastic isotropic solids

The constitutive model for linear elasticity is denoted Hooke's law. Originally the law was defined for a linear spring but generalized to an isotropic solid it reads as follows

$$\sigma_{ij} = \lambda\, \epsilon_{kk}\, \delta_{ij} + 2\mu\, \epsilon_{ij} \tag{4.12}$$

where ϵ_{ij} is the small strain tensor defined in (2.15). A small strain assumption has been made and it is therefore the small strain tensor can be used. This also means that the density ρ can be assumed to be approximately constant. The model parameters λ and μ are the Lame's constants, which are related to Young's modulus E and Poisson's ratio ν as follows

$$\lambda = \frac{E\, \nu}{(1 + \nu)\, (1 - 2\, \nu)}, \quad \mu = \frac{E}{2\, (1 + \nu)}$$

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